

University of Bath



**PHD**

## **Derivation of Kinetic Equations from Particle Models**

Stone, George Russell

*Award date:*  
2017

*Awarding institution:*  
University of Bath

[Link to publication](#)

### **General rights**

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal ?

### **Take down policy**

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Download date: 23. May. 2019

# Derivation of Kinetic Equations from Particle Models

submitted by

George Russell Stone

for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

September 2017

## **COPYRIGHT**

Attention is drawn to the fact that copyright of this thesis rests with the author. A copy of this thesis has been supplied on condition that anyone who consults it is understood to recognise that its copyright rests with the author and that they must not copy it or use material from it except as permitted by law or with the consent of the author.

This thesis may be made available for consultation within the University Library and may be photocopied or lent to other libraries for the purposes of consultation with effect from .....

Signed on behalf of the Faculty of Science .....

## Summary

The derivation of the Boltzmann equation from a particle model of a gas is currently a major area of research in mathematical physics. The standard approach to this problem is to study the BBGKY hierarchy, a system of equations that describe the distribution of the particles. A new method has recently been developed to tackle this problem by studying the probability of observing a specific history of events.

We further develop this method to derive the linear Boltzmann equation in the Boltzmann-Grad scaling from two similar Rayleigh gas hard-sphere particle models. In both models the initial distribution of the particles is random and their evolution is deterministic. Validity is shown up to arbitrarily large times and with only moderate moment assumptions on the non-equilibrium initial data.

The first model considers a Rayleigh gas whereby one tagged particle collides with a large number of background particles, which have no self interaction. The initial distribution of the background particles is assumed to be spatially homogeneous and at a collision between a background particle and the tagged particle only the tagged particle changes velocity.

In the second model we make two changes: we allow the background particles to have a spatially non-homogeneous initial data and we assume that at collision both the tagged particle and background particle change velocity.

The proof for each model follows the same general method, where we consider two evolution equations, the idealised and the empirical, on all possible collision histories. It is shown by a semigroup approach that there exists a solution to the idealised equation and that this solution is related to the solution of the linear Boltzmann equation. It is then shown that under the particle dynamics the distribution of collision histories solves the empirical equation. Convergence is shown by comparing the idealised and empirical equations.

# Acknowledgements

Firstly I would like to express my sincere appreciation to my supervisor, Dr Karsten Matthies, for the years of support and guidance and for devoting so much of your time to me. I am very grateful for all your assistance and your advice all the way back to when I first arrived at the university.

My thanks also extend to Dr Florian Theil for the feedback and suggestions on my work throughout my PhD.

I would like to thank my two viva examiners, Professor Johannes Zimmer and Professor Clément Mouhot, for their insightful comments, suggestions and corrections to the first version of this thesis.

I would like to thank EPSRC for providing the funding that allowed me to complete this PhD. In particular I am very grateful for the support that enabled me to travel to and present at a number of national and international conferences.

I would like to extend my gratitude to the entire staff and support network within the University of Bath for all the assistance and advice I have received.

I would like to take this opportunity to express my sincere gratitude to my entire family for the unwavering encouragement, endless support and for instilling within me a drive to achieve my dreams.

Finally I would like to thank all my mathematics teachers throughout my schooling years for developing my knowledge and passion for mathematics. Without this I would never have achieved so much.

# Publications

Chapter 2 and parts of Chapter 1 of this thesis are adapted from [35]:

Karsten Matthies, George Stone and Florian Theil. The derivation of the linear Boltzmann equation from a Rayleigh gas particle model. *Kinetic and Related Models* 11(2):8-8, 2018.

Chapter 3 of this thesis is adapted from [34]:

Karsten Matthies and George Stone. The derivation of a non-autonomous linear Boltzmann equation from a heterogeneous Rayleigh gas particle model. *Submitted for journal publication.* Preprint: <https://arxiv.org/abs/1706.03532>.

The work presented in this thesis is either wholly or substantially the original work of the named author.

# Contents

<b>1</b>	<b>Introduction</b>	<b>6</b>
1.1	Description of the Problem . . . . .	6
1.1.1	The Lorentz and Rayleigh Gases . . . . .	8
1.1.2	The Boltzmann Equation More Generally . . . . .	9
1.2	Model and Goals . . . . .	9
1.3	The BBGKY Hierarchy . . . . .	11
1.4	The Collision Tree Approach . . . . .	14
1.4.1	Tree Set Up . . . . .	15
1.4.2	The Idealised and Empirical Equations . . . . .	15
1.4.3	Convergence . . . . .	16
1.5	Semigroups and Honesty . . . . .	16
1.6	Main Results . . . . .	19
1.6.1	Outline of Method . . . . .	21
1.6.2	Comparison to [9] . . . . .	22
<b>2</b>	<b>The Autonomous Case</b>	<b>25</b>
2.1	Introduction . . . . .	25
2.2	Model and Main Result . . . . .	26
2.2.1	Remarks . . . . .	28
2.2.2	Method of Proof . . . . .	30
2.2.3	Collision Histories . . . . .	30
2.3	The Idealised Distribution . . . . .	32
2.4	The Empirical Distribution . . . . .	47
2.5	Convergence . . . . .	65
2.6	Proof of Auxiliary Results . . . . .	77
2.6.1	Particle dynamics . . . . .	77

<b>3</b>	<b>The Non-Autonomous Case</b>	<b>80</b>
3.1	Introduction . . . . .	80
3.2	Model and Main Result . . . . .	81
3.2.1	Remarks . . . . .	84
3.2.2	Method of Proof . . . . .	84
3.2.3	Collision Histories . . . . .	85
3.2.4	The Propagation of Chaos . . . . .	86
3.3	The Idealised Distribution . . . . .	87
3.3.1	The Evolution Semigroup . . . . .	89
3.3.2	Existence of Non-Autonomous Linear Boltzmann Solution . . . .	105
3.3.3	Building The Solution . . . . .	108
3.4	The Empirical Distribution . . . . .	126
3.4.1	Defining the Empirical Distribution . . . . .	127
3.4.2	Good Histories . . . . .	131
3.4.3	The Empirical Equation . . . . .	133
3.5	Convergence . . . . .	141
3.5.1	Comparing $P_t^\varepsilon$ and $\hat{P}_t^\varepsilon$ . . . . .	142
3.5.2	Convergence between $P_t^0$ and $\hat{P}_t^\varepsilon$ and the proof of theorem 3.2.4	148
	<b>Bibliography</b>	<b>155</b>

# Chapter 1

## Introduction

The derivation of continuum equations from atomistic particle dynamics is currently a major area of research in mathematical physics originating from Hilbert's Sixth Problem in 1900 with many questions still unsolved. Generally this has been approached in two steps: firstly by deriving kinetic equations, such as the Boltzmann equation, from particle mechanics and then secondly by deriving continuum equations such as Euler and Navier Stokes, from the Boltzmann equation.

We focus on the rigorous justification of the Boltzmann equation from a particle model of a gas. Recent results have continued the progress on this question but there is still much to be done.

### 1.1 Description of the Problem

The Boltzmann equation is a well known equation used in physics and mathematical modelling that can describe the evolution of a distribution of a dilute gas. The equation is given by,

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x F \cdot \nabla_v f &= Q(f, f), \\ f_{t=0} &= f_0, \end{cases}$$

where  $f = f_t(x, v)$  represents the distribution of the gas at position  $x$  and velocity  $v$  at time  $t$ ,  $F$  denotes the potential of an external force, the operator  $Q$  represents the effect of self-interaction amongst the particles and  $f_0$  is some given initial distribution. We study the equation in the absence of an external force, where  $F = 0$ .

The rigorous derivation and justification of the Boltzmann equation from a particle model of a gas is still an open question with several difficulties even though much research has been dedicated to it.

The research mostly focuses on two different particle models. In the first, known as



hard-spheres, particles are treated as billiard balls that collide via Newtonian mechanics, and in the second, known as short range potentials, the particles carry a force that effects other nearby particles. Although our research is focused on the first hard-sphere model, the second model is very similar and is still of interest.

We study the limiting behaviour of this model as the number of particles tends to infinity. As we increase the number of particles we decrease their radii in such a way that the expected number of collisions in a given time remains constant. This is known as the Boltzmann-Grad scaling. The goal of this area of research is to prove that the as the number of particles tends to infinity the probability of finding a particle at a given position converges to the solution of the Boltzmann equation.

The first major work in this area was [27]. This paper was able to show convergence from a hard-sphere particle model to the Boltzmann equation, via a hierarchy of equations known as the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy detailed below. The convergence was valid for short times, a fraction of the mean free flight time. This proof was simplified in [50] by employing Cauchy-Kowalevski arguments. Global in time convergence results were proved in [25, 41, 24] with the assumption of sufficiently large mean free paths.

An in depth overview of the BBGKY hierarchy, as well the Boltzmann equation more generally, can be found in [14].

A recent major work [20] collected many of the results in this area and was able to prove convergence for both hard-spheres and short range potentials. The authors were able to compute new estimates to help prove convergence. However this was again only valid for short times and well controlled initial data. The result in the short range potential case followed the ideas of [26]. Similar and more explicit results for short-range potentials were shown in [42].

In [9] the authors were able to utilise the tools from [20] to prove the convergence from a hard-sphere particle model to the linear Boltzmann equation with explicit convergence rates. The result holds for arbitrarily large times and for an initial distribution that is a small perturbation away from equilibrium for one particle and at equilibrium for all other particles. They were further able to use the linear Boltzmann equation as an intermediary step to prove convergence to Brownian motion. A similar method was adapted in [10] where the authors proved the convergence from a hard-sphere particle model, with an initial distribution where each particle is perturbed from equilibrium, to a linearised Boltzmann equation and then, under the right scaling, to the Stokes-Fourier equation.

A new method to tackle this problem has recently been developed in a series of papers [36, 37, 38]. This method employs semigroup techniques to study the conver-

gence, rather than the BBGKY hierarchy. This comes from studying the distribution of the history of the particles up to a certain time rather than the distribution of the particles at a specific time. These papers have been able to prove convergence for arbitrarily large times but only for a simplified particle interaction system. We detail this approach below.

The derivation of the Boltzmann equation from a system of particles interacting with long range potentials, where each particle effects every other particle regardless of their distance, has proved more difficult. One result was proved in the linear case via the BBGKY hierarchy with strong decay assumptions on the potential and for arbitrarily long times in [5].

### 1.1.1 The Lorentz and Rayleigh Gases

Instead of considering a system of a large number of identical hard-spheres evolving via elastic collisions one can consider a single tagged or tracer particle evolving among a system of fluid scatterers or background particles. With such a model one then considers the linear Boltzmann equation, where the operator  $Q$  now encodes the effect of the tagged particle interacting with the scatterers, rather than self interactions amongst the particles.

If the background particles are fixed and of infinite relative mass to the tracer particle then one has a model known as the Lorentz gas first introduced by Lorentz in [31] to study the motion of electrons in a metal.

Much research has been done deriving the linear Boltzmann equation from a Lorentz gas with randomly placed scatterers, for example [11, 22, 45, 21] and a large number of references found in [47, Part I Chapter 8]. The linear Boltzmann equation can however fail to hold if we consider non-random periodic scatterers, as shown for example in [23, 32]. The existence of a limiting stochastic process for the periodic Lorentz gas from the Boltzmann-Grad limit was shown in [33].

When a force field is present the convergence of the distribution of the tracer particle in an absorbing Lorentz gas to the solution of a gainless linear Boltzmann equation was proved in [17]. The authors also proved that if the scatterers move with a constant random initial velocity then the convergence can be proven with significantly weaker assumptions on the force field.

Closely related to the Lorentz gas is the Rayleigh gas, where the background particles are no longer of infinite mass. Convergence to Brownian motion is discussed in [47, Part I Chapter 8]. In [30] the convergence of the momentum process for a test particle to a jump process associated to the linear Boltzmann equation is proved. This holds for arbitrarily long times, via the BBGKY hierarchy, when the initial distribution of

the velocities is at equilibrium. This builds on their previous work [28, 29, 51].

In this thesis we consider two Rayleigh gas models where the background particles are of equal mass to the tagged particle and have no self interaction.

### 1.1.2 The Boltzmann Equation More Generally

As mentioned above, the derivation of the Boltzmann equation from particle dynamics is only the first step to approaching a solution to Hilbert's Sixth problem, with the second step being the derivation of continuum equations from the Boltzmann equation.

A comprehensive overview of the area is given in [44]. Under the appropriate scaling it is shown that the hydrodynamic limit of the Boltzmann equation gives the incompressible Navier-Stokes equation. With further assumptions, under the appropriate scaling, an incompressible Euler equation can be derived. At the end of the work the question of the compressible Euler equation is detailed as an open problem. A more brief overview of the problem is given by the same author in [43].

An excellent overview of the Boltzmann equation, including Cauchy theory for existence, long time convergence and a very wide range of citations is given in [19, Chapter 2].

Global in time existence to the Boltzmann equation was proved in [48, 49] when the initial data is a perturbation from equilibrium. The existence of a solution to the linear Boltzmann equation for more general initial data and a background at equilibrium is given by semigroup methods in [6, Chapter 10]. These methods are adapted to the non-autonomous linear Boltzmann equation in [4].

## 1.2 Model and Goals

We now describe the hard-sphere particle model and state the Boltzmann equation more formally. Let  $d \in \mathbb{N}$ . We consider particles evolving in some  $d$  dimensional domain denoted  $U$ . Usually it is assumed that  $U$  is the  $d$  dimensional unit torus, or  $U = \mathbb{R}^d$ . We consider  $N \in \mathbb{N}$  identical particles of equal unit mass and of diameter  $\varepsilon > 0$ , given by the Boltzmann-Grad scaling,  $N\varepsilon^{d-1} = 1$ . Let  $f_0 \in L^1(U \times \mathbb{R}^d)$ . Each particle has random and independent initial position and velocity given by  $f_0$ . Particles travel in free flow while they remain at least  $\varepsilon$  apart. When two particles come within  $\varepsilon$  they collide and change direction.

For  $1 \leq j \leq N$  and  $t \geq 0$  denote the position and velocity of particle  $j$  by  $(x_j(t), v_j(t))$ . Let  $1 \leq j \leq N$  and  $t \geq 0$ . Then

$$\frac{dx_j(t)}{dt} = v_j(t).$$

Now suppose that for all  $k \neq j$ ,  $|x_j(t) - x_k(t)| > \varepsilon$ , that is at time  $t$ , every other particle is at least  $\varepsilon$  away from particle  $j$ , then particle  $j$  has constant velocity, i.e.,

$$\frac{dv_j(t)}{dt} = 0.$$

Else there exists a  $k$  such that  $|x_j(t) - x_k(t)| = \varepsilon$  in which case particles  $j$  and  $k$  experience an instantaneous elastic collision at time  $t$ . Define the collision parameter  $\nu \in \mathbb{S}^{d-1}$ , where  $\mathbb{S}^k = \{x \in \mathbb{R}^{k+1}, |x| = 1\}$ , by

$$\nu := \frac{x_j(t) - x_k(t)}{|x_j(t) - x_k(t)|}.$$

Denoting the velocities before the collision as  $v_j(t^-)$  and  $v_k(t^-)$  and after the collision as  $v_j(t)$  and  $v_k(t)$  respectively we have that,

$$\begin{aligned} v_j(t) &= v_j(t^-) - \nu \cdot (v_j(t^-) - v_k(t^-))\nu, \\ v_k(t) &= v_k(t^-) + \nu \cdot (v_j(t^-) - v_k(t^-))\nu. \end{aligned}$$

These dynamics are well defined as long as 1) for any given time there is not a collision between at least three particles and 2) there is not infinitely many collisions within a finite time. We refer to [20, Proposition 4.1.1] for a proof that these pathological situations only occur for initial positions contained in a set of zero measure.

We remark that while the initial positions of the particles are random, the dynamics are entirely deterministic.

We now state the relevant Boltzmann equation for this hard-sphere particle model. Consider,

$$\begin{cases} \partial_t f_t + v \cdot \nabla_x f_t &= Q(f_t, f_t) \\ f_{t=0} &= f_0, \end{cases} \quad (1.2.1)$$

where the operator  $Q$ , representing the effects of collisions amongst the particles, is given by,

$$Q(f_t, f_t)(x, v) := \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^n} (f_t(x, v') f_t(x, \tilde{v}') - f_t(x, v) f_t(x, \tilde{v})) [(v - \tilde{v}) \cdot \nu]_+ d\tilde{v} d\nu, \quad (1.2.2)$$

where  $[y]_+ := \max\{y, 0\}$  and the pre-collisional velocities  $v'$  and  $\tilde{v}'$  are given by  $v' = v + \nu \cdot (\tilde{v} - v)\nu$  and  $\tilde{v}' = \tilde{v} - \nu \cdot (\tilde{v} - v)\nu$ .

We notice that  $Q(f_t, f_t)$  has two terms inside the integral. The first term represents

a particle colliding while travelling at velocity  $v'$  in such a way that it is now travelling at velocity  $v$  - a gain in the probability of finding a particle at velocity  $v$ . The second term represents a particle travelling at velocity  $v$  and colliding - a loss in the probability of finding a particle travelling at velocity  $v$ . We call these the gain and loss terms respectively.

The aims of this area of research can now be summarised as follows. Consider a system of  $N$  particles evolving via these dynamics and consider the distribution of finding a particle at any given position, velocity and time. We are aiming to show that as  $N$  tends to infinity, or equivalently as  $\varepsilon$  tends to zero, this distribution converges, in some sense, to the solution of the Boltzmann equation.

One fundamental difficulty in this problem is understanding molecular chaos, or the propagation of chaos, see [14, section 2.3]. To derive the Boltzmann equation we would like to say that our particle model has an independence property in that the probability of finding two particles about to collide is equal to the probability of finding one of the particles multiplied by the probability of finding the other. But unfortunately these are not always independent events since both particles could have previously collided with the same particle. The BBGKY hierarchy is a tool to better understand this issue.

### 1.3 The BBGKY Hierarchy

The classical approach to this problem is to consider the BBGKY hierarchy, as in for example [13, 14, 27, 47]. The BBGKY hierarchy is a system of  $N$  equations that describe the evolution of the particles. The  $s$ -th equation in the hierarchy describes the distribution of the first  $s$  particles and includes a term involving the distribution of the first  $(s + 1)$  particles. Each equation can be written as a sum and by comparing the terms of the sum with the solution of the Boltzmann equation we are able to prove convergence results.

We now refer specifically to [20], which covers both hard-spheres and short range repulsive potentials. For the hard-sphere case, [20] is a representation with minor improvements of [27]. Here  $U = \mathbb{R}^d$ . The main result in the hard-sphere case is [20, Chapter 3 Theorem 4],

**Theorem.** *Let  $f_0 : U \times \mathbb{R}^d \rightarrow \mathbb{R}_+$  be a continuous probability density such that,*

$$\left\| f_0(x, v) \exp \left( \frac{\beta}{2} |v|^2 \right) \right\|_{L^\infty(U \times \mathbb{R}^d)} < \infty,$$

*for some  $\beta > 0$ . Then, up to some time  $T$ , the distribution of  $N$  particles evolving via the hard-sphere dynamics described above converges in the sense of observables and in*

the Boltzmann-Grad limit,  $N\varepsilon^{d-1} = 1$ , to the solution of the Boltzmann equation with initial data  $f_0$ .

A more detailed statement, in terms of the BBGKY hierarchy, is stated in [20, Chapter 6 Theorem 8]. The corresponding short-range potential version of this theorem is given in [20, Chapter 3 Theorem 5].

We now describe the set up and method of proof. Consider  $N$  particles each with position and velocity  $z_j = (x_j, v_j)$  evolving via the hard-sphere particle dynamics described above. Define  $Z_N = (z_1, \dots, z_N)$  and

$$\mathcal{D}_N := \left\{ Z_N \in (U \times \mathbb{R}^d)^N : \text{for } i \neq j, |x_i - x_j| > \varepsilon \right\}. \quad (1.3.1)$$

That is  $\mathcal{D}_N$  denotes the space where particles travel in free flow. For  $t \geq 0$  let  $f_N(t)$  denote the joint distribution of the  $N$  particles at time  $t$ . It follows by [20, section 1.1], that

$$\partial_t f_N(t) + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N(t) = 0 \text{ on } \mathcal{D}_N.$$

This is the Liouville equation, with boundary conditions that for any precollisional  $N$  particle configuration  $Z_N^{in}$  and its corresponding post collisional configuration  $Z_N^{out}$ ,  $f_N(Z_N^{in}) = f_N(Z_N^{out})$ . For  $1 \leq s \leq N$  define  $f_N^{(s)}(t)$ , the  $s$  particle marginal of  $f_N(t)$ , by

$$f_N^{(s)}(t, Z_s) := \int_{(U \times \mathbb{R}^d)^{N-s}} f_N(t, Z_s, z_{s+1}, \dots, z_N) \mathbb{1}_{Z_N \in \mathcal{D}_N} dz_{s+1} \dots dz_N.$$

Then  $f_N^{(1)}(t, z_1)$  represents the probability of finding a particle at  $z_1$  at time  $t$ . Since the particles are indistinguishable but not independent it follows that  $f_N^{(1)}(t)$  depends on  $f_N^{(2)}(t)$ . By considering the Liouville equation in weak form and the boundary conditions on the distributions, as in [20, section 4.3], it can be shown that, for each  $1 \leq s \leq N$  and for any  $t \geq 0$ ,

$$\partial_t f_N^{(s)}(t) + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N^{(s)}(t) = \mathcal{C}_{s,s+1} f_N^{(s+1)}(t) \text{ on } \mathcal{D}_s,$$

where  $\mathcal{C}_{s,s+1}$ , detailed in [20, section 4.3], represents the effect of collisions between one of the  $s$  particles and another particle. This is the BBGKY hierarchy. Now define  $f_{0,N}^{(N)}(Z_N) := \mathcal{Z}_N^{-1} \mathbb{1}_{Z_N \in \mathcal{D}_N} f_0^{\otimes N}(Z_N)$ , where  $\mathcal{Z}_N$  is a normalising constant so that  $f_{0,N}^{(N)}$  has unit mass. Further define  $f_{0,N}^{(s)}$  to be the  $s$ -th marginal of  $f_{0,N}^{(N)}$ . Integrating the

BBGKY hierarchy gives,

$$f_N^{(s)}(t, Z_s) = T_s(t)f_N^{(s)}(0, Z_s) + \int_0^t T_s(t-\tau)\mathcal{C}_{s,s+1}f_N^{(s+1)}(\tau, Z_s) d\tau,$$

with boundary conditions  $f_s(Z_s^{in}) = f_s(Z_s^{out})$  for any precollisional  $s$  particle configuration  $Z_s^{in}$  with corresponding postcollisional  $s$  particle configuration  $Z_s^{out}$ , where  $T_s(t)$ , as described in [20, (4.3.6)], denotes the solution operator of the  $s$  particle flow. We now consider the asymptotic dynamics, at least formally. Denoting the asymptotic limit of  $\mathcal{C}_{s,s+1}$  by  $\mathcal{C}_{s,s+1}^0$ ,  $T_s$  by  $S_s$  and  $f_N^{(s)}$  by  $f^{(s)}$  we have that

$$f^{(s)}(t, Z_s) = S_s(t)f_0^{(s)}(Z_s) + \int_0^t S_s(t-\tau)\mathcal{C}_{s,s+1}^0f^{(s+1)}(\tau, Z_s) d\tau.$$

This is known as the Boltzmann hierarchy. The existence of mild solutions to the BBGKY and Boltzmann hierarchy is covered in [20, Theorem 6 and 7] respectively. Uniform bounds on the solutions are also derived, which is the cause of the time restriction. The BBGKY and Boltzmann hierarchy can be written in terms of the initial data as,

$$\begin{aligned} f_N^{(s)}(t) = & \sum_{k=0}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} T_s(t-t_1)\mathcal{C}_{s,s+1}T_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2} \\ & \cdots T_{s+k}(t_k)f_{0,N}^{(s+k)} dt_k \cdots dt_1, \end{aligned} \quad (1.3.2)$$

with the convention that  $f_N^{(s)} = 0$  if  $j > N$ , and,

$$\begin{aligned} f^{(s)}(t) = & \sum_{k=0}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} S_s(t-t_1)\mathcal{C}_{s,s+1}^0S_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}^0 \\ & \cdots S_{s+k}(t_k)f_0^{(s+k)} dt_k \cdots dt_1. \end{aligned} \quad (1.3.3)$$

To prove the main theorem we prove convergence between the terms inside the sums of (1.3.2) and (1.3.3). This is done by

- considering only a finite number of collisions,
- conditioning on bounded kinetic energy,
- conditioning on sufficiently separated collision times,
- demonstrating that re-collisions only happen for a small set of pathological trajectories,

- removing a small set of re-collision possibilities,
- approximating the initial data in the BBGKY hierarchy with the initial data from the Boltzmann hierarchy
- approximating the limit of coefficients in the Boltzmann-Grad scaling .

Explicit convergence estimates for each of these steps are covered in [20, Chapters 7, 12 and 14]. Convergence can then be shown by chaining each of these estimates together. By using that  $f^{(1)}$  solves the Boltzmann equation this gives in particular that  $f_N^{(1)}$ , the distribution of finding a particle in the  $N$  particle hard-sphere flow, converges to the solutions of the Boltzmann equation. The restriction in time results from the restriction in time for the existence of solutions to the hierarchies with uniform bounds.

The proof for short range potentials follows similarly.

## 1.4 The Collision Tree Approach

The second approach to this problem, which we call the collision tree approach, arises from considering the probability of observing a specific history of collisions up to a certain time, rather than considering the distribution of the particles at a specific time. This is the approach that we develop in the later chapters of this thesis.

We refer specifically to [38], which is a continuation of [37, 36]. The result is shown for an annihilation particle model, a simplification of the the hard-sphere model, whereby particles are removed after a collision. Because of this the Boltzmann equation that we are aiming to prove is a modification of (1.2.1) where the gain term of (1.2.2) is removed - this corresponds to the fact that in the model if the particle changes velocity then it has experienced a collision and so no longer exists. For  $1 \leq j \leq N$  define the scattering state of particle  $j$ ,  $\beta_j^{(\varepsilon)}(t)$  to be 1 if particle  $j$  has not experienced any collisions up to time  $t$  or 0 otherwise.

The main result, [38, Theorem 2.1], is as follows,

**Theorem.** *Let  $0 < T < \infty$ ,  $f_0$  be such that [38, (1.3) - (1.5)] hold and  $d \geq 2$ . Then for any  $\delta > 0$  and any  $A \subset U \times \mathbb{R}^d$  open, uniformly for  $t \in [0, T]$ ,*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P} \left( \left| \frac{1}{N} \# \left\{ j : (x_j(t), v_j(t)) \in A, \beta_j^{(\varepsilon)}(t) = 1 \right\} - \int_A f_t(x, v) \, dx \, dv \right| > \delta \right) = 0,$$

where  $f_t$  is the mild solution of the gainless Boltzmann equation.

The merit of this method is that result holds up to an arbitrarily large time and with only finite moment assumptions on the initial data  $f_0$ .



The result is proved by defining the concept of collision histories, or trees, that encode the state of the system up to a given time. Two Kolmogorov equations on trees are considered. The first, the idealised, can be thought of a tree version of the Boltzmann equation, and the second, the empirical, relates to the particle dynamics. The main theorem is proven as a corollary of the convergence between the solutions of these two Kolmogorov equations.

#### 1.4.1 Tree Set Up

We introduce an intermediary step to prove convergence by defining collision trees. In these trees we assume that the particles behave as hard-spheres and collide without being removed, with the annihilation aspect of the model added later.

A rooted, non-cyclic tree represents a specific history of collisions effecting a given particle. Each node represents a particle and each edge a collision between the particles it connects. The root of the tree is marked with  $(x_0, v_0)$  the initial position and velocity of the particle. Each child node of the root represents the particles that the root collides with and are marked with  $(s_l, \nu_l, v_l) \in [0, T] \times \mathbb{S}^{d-1} \times \mathbb{R}^d$  the collision time, collision parameter and incoming velocity of the other particle in the collision. Each of these child nodes of the root has its own child nodes which represent the particles that this particle collides with in  $[0, s_l]$  and are again marked with the collision time, parameter and incoming velocity.

Denote the set of all marked trees by  $\mathcal{MT}$ . For a tree  $\Phi \in \mathcal{MT}$  we introduce the notation  $(\tau, \nu, v')$  to denote the marker representing the final collision of the root. Denote by  $\Phi'$  the tree including the child node representing the final collision of  $\Phi$ , and all child nodes of this node. Denote  $\bar{\Phi}$  to be  $\Phi$  with  $\Phi'$  removed. We define  $m(\Phi)$  to be the set of nodes in  $\Phi$ .

For a more detailed explanation of collision trees see [38, Chapter 3], or section 2.2.3 in this thesis.

#### 1.4.2 The Idealised and Empirical Equations

The idealised equation represents a tree version of the Boltzmann equation. We first state the equation and then prove that there exists a solution. The idealised equation is given by,

$$\begin{cases} \partial_t P_t^\varepsilon(\Phi) &= \mathcal{Q}_t^\varepsilon[P_t^\varepsilon](\Phi) \\ P_0^\varepsilon(\Phi) &= \mathbb{1}_{\#m(\Phi)=1} f_0(x_0, v_0), \end{cases}$$

where  $\mathcal{Q}_t^\varepsilon[P](\Phi) = \mathcal{Q}_{t,+}^\varepsilon[P](\Phi) - \mathcal{Q}_{t,-}^\varepsilon[P](\Phi)$  and,

$$\begin{aligned}\mathcal{Q}_{t,+}^\varepsilon[P](\Phi) &= \delta(t - \tau) P(\bar{\Phi}) P(\Phi') [(v - v') \cdot \nu]_+ \\ \mathcal{Q}_{t,-}^\varepsilon[P](\Phi) &= P(\Phi) \int_{\mathbb{S}^{n-1}} \int_{\mathcal{MT}} \delta(x_0 - x_0(\Psi) + t(v_0 - v_0(\Psi)) + \varepsilon \nu') \\ &\quad [(v - v(\Psi)) \cdot \nu']_+ dP(\Psi) d\nu'.\end{aligned}$$

We remark that in the loss rate  $\mathcal{Q}_{t,-}^\varepsilon[P](\Phi)$  we are integrating over all trees  $\Psi$  using the measure  $P$  on  $\mathcal{MT}$ . The existence of a solution is proven by employing evolution semigroup techniques, specifically the results of [40, Chapter 5]. It is then proven that  $P_t^\varepsilon$  converges to  $P_t^0$  and that integrating  $P_t^0$  over a specific set of collision trees gives the solution of the gainless Boltzmann equation.

Now consider  $\hat{P}_t^\varepsilon$ , the probability distribution on  $\mathcal{MT}$  defined by the annihilation particle dynamics. It is shown by explicit calculation that, at least for well controlled or ‘good’ trees,  $\hat{P}_t^\varepsilon$  solves the empirical equation,

$$\begin{cases} \partial_t \hat{P}_t^\varepsilon(\Phi) &= (1 - \gamma(\Phi)) \hat{\mathcal{Q}}_t^\varepsilon[\hat{P}_t^\varepsilon](\Phi) \\ \hat{P}_0^\varepsilon(\Phi) &= \zeta(\Phi) \mathbb{1}_{\#m(\Phi)=1} f_0(x_0, v_0), \end{cases}$$

where  $\gamma(\Phi) = \#m(\Phi)\varepsilon^{d-1}$ ,  $\zeta(\Phi)$ , given explicitly in [38, (5.5)], excludes the effect of initial overlap and  $\hat{\mathcal{Q}}_t^\varepsilon$  is similar to  $\mathcal{Q}_t^\varepsilon$  but includes additional complications from the particle dynamics.

### 1.4.3 Convergence

After proving that there exists a solution  $P_t^\varepsilon$  to the idealised equation and that  $\hat{P}_t^\varepsilon$  solves the empirical equation for good trees, we then prove convergence in total variation between  $\hat{P}_t^\varepsilon$  and  $P_t^0$ . This is done by showing that the contribution from trees that are not good vanishes and then exploiting the similarity of the idealised and empirical equations.

Since  $P_t^0$  is connected to the solution of the Boltzmann equation and  $\hat{P}_t^\varepsilon$  is connected to the particle dynamics, the proof of the main theorem then follows.

## 1.5 Semigroups and Honesty

In Chapter 2 of this thesis we adapt theory from [6, Chapter 10] to prove the existence of a solution to the relevant autonomous linear Boltzmann equation.

We note that we were unable to apply standard existence results to our situation.

For example [18] is in the spatially homogeneous case, we were unable to verify the conditions of [16, section 21.3] and our gain and loss operators are not bounded functions as required in [1, theorem 3.1.2].

We now present a brief overview of the main ideas of [6]. Consider the autonomous Cauchy problem,

$$\begin{cases} \partial_t f_t &= A f_t + B f_t, \\ f_{t=0} &= f_0 \end{cases}$$

for operators  $A$  and  $B$ , with  $A$  generating a strongly continuous ( $C_0$ ) semigroup and  $B$  being positive, and some initial data  $f_0$ . Suppose there exists a semigroup  $G(t)$  solving this Cauchy problem. Then  $G(t)$  is a solution of a reformulation of the problem where the right hand side is the generator of the semigroup  $G(t)$ , denoted  $K$ . The issue is that  $K$  is unknown and the solution defined by the semigroup  $G(t)$  may behave differently to the behaviour expected from the operators  $A$  and  $B$ .

Thus when considering semigroup solutions to this problem we are required to show that the semigroup is honest, that is the semigroup behaves in the expected way, in order for the solution to make sense as a solution to the original problem.

We now describe this in more detail. Let  $X = L^1(\Omega)$ . A semigroup  $G(t)$  on  $X$  is called a substochastic semigroup if  $\|G(t)f\| \leq \|f\|$  and  $G(t)f \geq 0$  for all  $f \geq 0$  and  $t \geq 0$ . It is called stochastic if  $\|G(t)f\| = \|f\|$  and  $G(t)f \geq 0$  for all  $f \geq 0$  and  $t \geq 0$ . The semigroup  $G(t)$  is called a contraction if  $\|G(t)f\| \leq \|f\|$  for all  $f \in X$  and all  $t \geq 0$ .

**Corollary** ([6] Corollary 5.17). *Suppose that the operators  $A$  and  $B$  satisfy,*

1.  $(A, D(A))$  generates a substochastic semigroup  $G_A(t)$ ,
2.  $D(A) \subset D(B)$  and  $Bf \geq 0$  for all  $f \in D(B)_+$ ,
3. for all  $f \in D(A)_+$ ,

$$\int_{\Omega} Af + Bf \, d\mu \leq 0.$$

*then the assumptions of [6, Theorem 5.2] hold.*

The conclusion of [6, Theorem 5.2] is that there exists an extension  $(K, D(K))$  of  $(A+B, D(A))$  generating a contraction semigroup  $G_K(t)$ . Now assume that the system is conservative, that is, for all  $f \in D(A)$ ,

$$\int_{\Omega} Af + Bf \, d\mu = 0.$$

This is indeed true for the linear Boltzmann equation in our case, representing the fact that the total lost rate is equal to the total gain rate. Then if  $D(K) = D(A)$  or

$\overline{D(A)}$  the solution  $f(t) = G_K(t)f_0$  retains this conservative property and so semigroup solutions behave in the expected way. However if  $D(K)$  is strictly larger than  $\overline{D(A)}$  then the solution  $f(t) = G_K(t)f_0$  may not retain this conservative property everywhere - in such a case the solution decays mass and so does not correspond to a solution of the original system. This defines the idea of honesty, which we now define rigorously, at least in the conservative case.

**Definition.** The positive semigroup  $G_K(t)$  generated by  $K$ , an extension of  $A + B$ , is *honest* if for any  $f_0 \in D(K)$  the solution  $f(t) = G_K(t)f_0$  satisfies

$$\frac{d}{dt} \int_{\Omega} f(t) d\mu = 0.$$

In fact in the conservative case honesty is equivalent to the semigroup  $G_K(t)$  being stochastic [6, Proposition 6.9].

Above we remarked that mass is conserved if  $D(K)$  is equal to  $D(A)$  or  $\overline{D(A)}$ . In fact this is also sufficient,

**Theorem** ([6] Theorem 6.22).  $G_K(t)$  is honest if and only if  $K = \overline{A + B}$ .

In the case of the autonomous linear Boltzmann equation, the operator  $A$  includes the effect of the transport and loss term and  $B$  the effect of the gain term. In [6, theorem 10.28] it is shown that under assumptions (A1) - (A7) [6, Chapter 10],  $K = \overline{A + B}$  and hence,  $G_K(t)$  is honest by the above result. In Chapter 2 of this thesis, assumptions (A1) - (A6) are straightforward properties of the model but (A7), a bound on the effect of  $B$ , requires some calculation.

In order to prove that (A7) holds we are required to assume that the initial distribution of the background particles has finite fifth  $L^\infty$  moment. The set of background distributions where honesty does not hold is not clear.

In Chapter 3 of this thesis we adapt some of the results from [4], which considers existence and honesty theory for the linear Boltzmann equation in the non-autonomous case. We use [4] for existence results but were unable to adapt the honesty results to our situation. Instead honesty is proved more explicitly by showing that the semigroup indeed preserves mass.

We remark that in chapter 2 we could use the same direct argument to prove honesty as in chapter 3 by calculating the mass of the solution. This would also allow us to reduce the required power in the  $L^\infty$  assumption of the initial data. However the method of using [6] as described above is significantly shorter, more efficient and more elegant. Indeed compare the honesty arguments in chapter 2, lemma 2.3.7 and

proposition 2.3.4, to all the technical results needed in section 3.3.3 to prove honesty in chapter 3.

## 1.6 Main Results

In this thesis we continue the collision tree approach to the derivation of the Boltzmann equation. By following this method we derive the linear Boltzmann equation from two Rayleigh gas particle models. In both cases validity is shown for arbitrarily large times and with finite moment assumptions on the initial data.

In Chapter 2 we assume that the distribution of the background particles is spatially homogeneous, that the background particles have no self-interaction and that, at a collision between the tagged particle and a background particle, the background particle does not change velocity.

In Chapter 3 we drop the assumption that the initial distribution of the background particles is spatially homogeneous and also assume that at a collision between the tagged particle and a background particles there is a full hard-sphere collision and both particles change velocity.

For each model we consider  $d = 3$  and  $U = \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$  the flat three dimensional unit torus. If we denote the distribution of the tagged particle at time  $t$ , evolving via the relevant Rayleigh gas dynamics, as  $\hat{f}_t^N$ , then in both models we prove that  $\hat{f}_t^N$  converges in total variation to  $f_t$ , the solution of the relevant linear Boltzmann equation, uniformly for  $t$  in a compact set.

We closely follow [38], considering two Kolmogorov equations on trees. However in both chapters 2 and 3, because the background particles do not interact with each other, the tree structure is simplified to just a list of collisions.

We first construct a solution to the idealised equation and show the connection between this solution and the solution of the relevant linear Boltzmann equation. We then show that the empirical distribution on trees resulting from the particle dynamics solves a similar equation. By proving the convergence between the idealised and empirical solutions the required convergence between  $\hat{f}_t^N$  and  $f_t$  follows.

While the method for both chapters is the same, Chapter 3 has the significant additional difficulty that the relevant linear Boltzmann equation is non-autonomous due to the fact that in this model the initial distribution of the background particles is spatially heterogeneous. This requires more technical evolution semigroup results. We believe that Chapter 3 represents the first derivation of a non-autonomous linear Boltzmann equation.

We now state the two main theorems of this thesis. Firstly in chapter 2 we prove:

**Theorem** (Theorem 2.2.4). *Let  $0 < T < \infty$  and suppose that the initial distribution of the tagged particle  $f_0$  and the initial distribution of the background particles  $g_0$  satisfies the finite moment assumptions in definition 2.2.2. Then uniformly for  $t \in [0, T]$   $\hat{f}_t^N$ , the distribution of the tagged particle at time  $t$  evolving via  $N$  background particles by the dynamics described in section 2.2, converges to the time dependent density  $f_t$  in total variation as  $N$  tends to infinity. Moreover the limit  $f_t$  satisfies the linear Boltzmann equation,*

$$\begin{cases} \partial_t f_t(x, v) &= -v \cdot \nabla_x f_t(x, v) + Q[f_t](x, v), \\ f_{t=0}(x, v) &= f_0(x, v), \end{cases}$$

where the collision operator  $Q$  is defined by  $Q := Q^+ - Q^-$  and  $Q^+$  and  $Q^-$  known respectively as the gain and loss term are given as follows,

$$Q^+[f](x, v) = \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} f(x, v') g_0(\bar{v}') [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu,$$

where the pre-collision velocities,  $v'$  and  $\bar{v}'$ , are given by  $v' = v + \nu \cdot (\bar{v} - v)\nu$  and  $\bar{v}' = \bar{v} - \nu \cdot (\bar{v} - v)\nu$ , and

$$Q^-[f](x, v) = f(x, v) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu.$$

The main result of chapter 3 is,

**Theorem** (Theorem 3.2.4). *Let  $0 < T < \infty$  and suppose that the initial distribution of the tagged particle  $f_0$  and the initial distribution of the background particles  $g_0$  satisfies the moment and regularity assumptions of definition 3.2.3. Then uniformly for  $t \in [0, T]$   $\hat{f}_t^N$ , the distribution of the tagged particle at time  $t$  evolving among  $N$  background particles via the dynamics described in section 3.2, converges in total variation to  $f_t^0$  as  $N$  tends to infinity where  $f_t^0$  is a solution of the following non-autonomous linear Boltzmann equation,*

$$\begin{cases} \partial_t f_t^0(x, v) &= -v \cdot \nabla_x f_t^0(x, v) + Q_t^0[f_t^0](x, v), \\ f_{t=0}^0(x, v) &= f_0(x, v). \end{cases}$$

Where  $Q_t^0 := Q_t^{0,+} - Q_t^{0,-}$  which are given by

$$Q_t^{0,+}[f](x, v) := \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} f(x, v') g_t(x, \bar{v}') [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu,$$

and,

$$Q_t^{0,-}[f](x,v) := f(x,v) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x,\bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu,$$

where we use the notation  $g_t(x,v) := g_0(x - tv, v)$  and where the pre-collision velocities,  $v'$  and  $\bar{v}'$ , are given by  $v' = v + \nu \cdot (\bar{v} - v)\nu$  and  $\bar{v}' = \bar{v} - \nu \cdot (\bar{v} - v)\nu$ .

### 1.6.1 Outline of Method

The collision tree method we employ for both chapter 2 and 3, which closely follows [38], is readily represented by the following diagram:

$$\begin{array}{ccc} \hat{f}_t^N & \xrightarrow{(1)} & f_t \\ \updownarrow (3) & & \updownarrow (2) \\ \hat{P}_t^\varepsilon & \xrightarrow{(4)} & P_t \end{array}$$

In the main theorems of each chapter we are aiming to prove the convergence represented by the arrow numbered (1) as  $N$  tends to infinity, that is that  $\hat{f}_t^N$ , the distribution of the tagged particle evolving among  $N$  background particles, converges to  $f_t$  the solution of the relevant linear Boltzmann equation.

Firstly we show in propositions 2.3.4 and 3.3.16 that there exists a solution to the relevant linear Boltzmann equation. Then we define the idealised equation, an evolution equation defined on all possible collision histories. The idealised equation can be thought of as equivalent to the linear Boltzmann equation on the space of all possible collision histories rather than the space of positions and velocities. In theorems 2.3.1 and 3.3.1 we show that the idealised equation has a solution which we denote by  $P_t$ . These theorems also prove the connection (2) between  $f_t$  and  $P_t$  by showing that  $f_t$  can be written as a marginal of  $P_t$  when integrated over an appropriate set of collision histories.

Now for each possible collision history  $\Phi$  let  $\hat{P}_t^\varepsilon(\Phi)$  denote the probability that the  $\Phi$  exists at time  $t$  when observing the tagged particle evolving among  $N$  background particles as described by the respective particle dynamics. Then the connection (3) between  $\hat{f}_t^N$  and  $\hat{P}_t^\varepsilon$  can be seen by again considering the marginal of  $\hat{P}_t^\varepsilon$  when integrated over an appropriate set of collision trees. It is then shown in theorems 2.4.6 and 3.4.17 that, at least when considering well controlled trees or ‘good’ trees,  $\hat{P}_t^\varepsilon$  satisfies an evolution equation, which we call the empirical equation.

Then the convergence represented by arrow (4) in the diagram between  $\hat{P}_t^\varepsilon$  and  $P_t$  is shown in theorems 2.5.8 and 3.5.6. This is proven by exploiting the similarity between the idealised and empirical equations and by using the fact that good histories have full measure in the limit.

Finally the proof of the main theorems of chapter 2 and 3, the convergence (1) between  $\hat{f}_t^N$  and  $f_t$ , follows from the connections (2) and (3) and using the convergence in (4).

### 1.6.2 Comparison to [9]

Of particular similarity to the main results of this thesis is [9]. We now give a description of the main results of [9] followed by a comparison to our results. In [9] a linear Boltzmann equation is derived from a hard sphere particle model via the BBGKY hierarchy for long times assuming an initial distribution of the particles which is a perturbation away from equilibrium. Then convergence to the diffusion equation and a Brownian motion is proved under the appropriate scaling.

Consider a system of  $N$  identical particles with positions and velocities given by  $(x_1, v_1), \dots, (x_N, v_N)$  evolving via hard sphere dynamics on the  $d$  dimensional torus  $\mathbb{T}^d$  for  $d \geq 2$ . For  $1 \leq s \leq N$  define  $Z_s = ((x_1, v_1), \dots, (x_s, v_s))$  and  $V_s = (v_1, \dots, v_s)$ . Let  $\beta > 0$ , define the Maxwellian distribution,

$$M_\beta(v) := \left(\frac{\beta}{2\pi}\right)^{d/2} \exp\left(-\frac{\beta}{2}|v|^2\right),$$

and

$$M_\beta^{\otimes s}(V_s) := \prod_{i=1}^s M_\beta(v_i).$$

Define the Hamiltonian,

$$H_N(V_N) := \frac{1}{2} \sum_{i=1}^N |v_i|^2,$$

and the Gibbs measure

$$M_{N,\beta}(Z_N) := \frac{1}{\mathcal{Z}_N} \mathbb{1}_{\mathcal{D}_N}(Z_N) M_\beta^{\otimes N}(V_N)$$

with  $\mathcal{D}_N$  is as in (1.3.1) and  $\mathcal{Z}_N$  is a normalising constant so that the measure has unit mass.

Now consider one tagged particle labelled 1 with position and velocity  $(x_1, v_1)$  and let  $\rho^0$  be a continuous probability measure on  $\mathbb{T}^d$ . Define a probability measure on  $(\mathbb{T}^d \times \mathbb{R}^d)^N$  by,

$$f_N^0(Z_N) := M_{N,\beta}(Z_N) \rho^0(x_1).$$

Note that this is a perturbation away from equilibrium for only the position component



of particle 1. For  $\alpha > 1$  consider the linear Boltzmann equation,

$$\begin{cases} \partial_t \varphi_\alpha + v \cdot \nabla_x \varphi_\alpha &= -\alpha \mathcal{L} \varphi_\alpha \\ \varphi_\alpha(t=0) &= \rho^0 \end{cases} \quad (1.6.1)$$

where

$$\mathcal{L} \varphi_\alpha(v) := \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} (\varphi_\alpha(v) - \varphi_\alpha(v')) M_\beta(v_1) [(v - v_1) \cdot \nu]_+ dv_1 d\nu,$$

with  $v' = v + \nu(v_1 - v)\nu$ . The main theorem of [9] can now be stated.

**Theorem** ([9] theorem 2.2). *Consider a system of  $N$  hard sphere particles with initial distribution  $f_N^0$ . Then the distribution of the tagged particle  $f_N^{(1)}(t, x, v)$  converges to  $M_\beta(v) \varphi_\alpha(t, x, v)$ , where  $\varphi_\alpha$  is the solution (1.6.1). More precisely for all  $t > 0$  and  $\alpha > 1$  in the limit  $N$  tending to  $\infty$  with  $N\varepsilon^{d-1} = \alpha$  one has,*

$$\|f_N^{(1)}(t, x, v) - M_\beta(v) \varphi_\alpha(t, x, v)\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq C \left( \frac{t\alpha}{(\log \log N)^{\frac{A-1}{A}}} \right)^{\frac{A^2}{A-1}} \quad (1.6.2)$$

where  $A \geq 2$  can be taken arbitrarily large and  $C$  depends on  $A, \beta, d$  and  $\|\rho^0\|_{L^\infty}$ .

The proof of this result uses the BBGKY approach and is similar to the approach of [20] discussed above.

Now define the time scaled trajectory of the tagged particle  $\Xi(\tau) := x_1(\alpha\tau)$  and note that  $\Xi(\tau)$  is defined by  $f_N^{(1)}(\alpha\tau, x, v)$ . The second main result of [9] is the following, which proves convergence to the heat equation and Brownian motion.

**Theorem** ([9] theorem 2.3). *Consider a system of  $N$  hard sphere particles with initial distribution  $f_N^0$  and assume  $\rho^0 \in C^0(\mathbb{T}^d)$ . Let  $\rho(\tau, x)$  denote the solution of the heat equation,*

$$\begin{cases} \partial_\tau \rho - \kappa_\beta \Delta \rho &= 0 \\ \rho(\tau=0) &= \rho^0. \end{cases}$$

Then,

$$\|f_N^{(1)}(\alpha\tau, x, v) - \rho(\tau, x) M_\beta(v)\|_{L^\infty([0, T] \times \mathbb{T}^d \times \mathbb{R}^d)} \rightarrow 0 \quad (1.6.3)$$

as  $N$  tends to infinity with  $\alpha = N\varepsilon^{d-1}$  going to infinity much slower than  $\sqrt{\log \log N}$ .

Moreover in the same limit  $\Xi(\tau)$  converges in law towards a Brownian motion of variance  $\kappa_\beta$  initially distributed under the measure  $\rho^0$ .

We now compare these results to the results presented here in this thesis. The significant difference in the methods is that we use the collision history method explained above, with [9] using the BBGKY hierarchy.

- In [9] particles are modelled as a full hard sphere system whereas we consider a simplified particle model where the background particles ignore each other and in chapter 2 the background particles do not even react at a collision.
- In our results we first fix an arbitrary  $T > 0$  and then prove convergence up to this time  $T$  without any explicit converge rates. However in [9] an explicit bound is proved between the distribution of the tagged particle and the solution of the linear Boltzmann equation for any time  $t$  and for any  $N$  which converges to zero as  $N$  tends to infinity.
- In [9] the convergence to the linear Boltzmann equation is used to prove the convergence to the heat equation and Brownian motion. We have not considered this convergence but the results of [9] suggests that this should be possible in our case.
- As for the initial data, in [9] it is assumed that initially all particles are at equilibrium and there is a perturbation in the position of the tagged particle  $\rho^0$ . However in our results we assume some finite moments and some regularity assumptions on the initial distribution of the tagged particle  $f_0$  and the background particles  $g_0$ , with chapter 2 assuming that the background particles are initially spatially homogeneous.

## Chapter 2

# The Autonomous Case

### 2.1 Introduction

In this chapter we derive a linear Boltzmann equation in the Boltzmann-Grad limit from a Rayleigh gas particle model where the background particles are of equal mass to the tagged particle and have no self interaction.

The particles evolve via a simplified form of hard-sphere dynamics whereby the background particles do not change velocity even at a collision. We consider non-equilibrium initial data but require that the background particles are spatially homogeneous. Convergence is proved for arbitrary large times and with finite moment assumptions on the initial data via the semigroup approach of [38] discussed in section 1.4.

The main result is theorem 2.2.4, which proves that the distribution of the tagged particle evolving among  $N$  background particles converges to the solution of the linear Boltzmann equation in total variation as the number of particles converges to infinity. This is shown by considering two equations, the idealised and the empirical, on the set of collision trees.

In section 2.3 the idealised equation is shown to have a solution via semigroup techniques and this solution is then shown to be related to the Boltzmann equation. Then in section 2.4 it is shown that the particle dynamics leads to a solution of the empirical equation. In section 2.5 it is shown that the solution of the empirical equation converges to the solution of the idealised equation, and from this main theorem follows.

## 2.2 Model and Main Result

Our Rayleigh gas model in three dimensional space is now detailed. Define  $U := \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$  to be the flat three dimensional unit torus. Here a tagged particle evolves via the hard sphere flow and the remaining  $N$  particles do not interact, i.e. move along straight lines. The initial distribution of the tagged particle is  $f_0 \in L^1(U \times \mathbb{R}^3)$ . The  $N$  background particles are independently distributed according to the law  $g_0 \in L^1(\mathbb{R}^3)$  in velocity space and uniform in  $U$ .

The tagged particle and the background particles are spheres with diameter  $\varepsilon > 0$  which is related to  $N$  via the Boltzmann-Grad scaling,

$$N\varepsilon^2 = 1. \tag{2.2.1}$$

The background particles always travel in free flow with their velocities never changing from the initial value. The tagged particle travels in free flow whilst its centre remains at least  $\varepsilon$  away from the centre of all the background particles.

When the centre of the tagged particle comes within  $\varepsilon$  of the centre of a background particle the tagged particle collides as if it was a Newtonian hard-sphere collision and changes velocity.

Explicitly this is described as follows. Denote the position and velocity of background particle  $1 \leq j \leq N$  at time  $t$  by  $(x_j(t), v_j(t))$ . Then for all  $t \geq 0$ ,

$$\frac{dx_j(t)}{dt} = v_j(t) \text{ and } \frac{dv_j(t)}{dt} = 0.$$

Further denote the position and velocity of the tagged particle at time  $t$  by  $(x(t), v(t))$ . Then for all  $t \geq 0$ ,

$$\frac{dx(t)}{dt} = v(t).$$

If at time  $t$  for all  $1 \leq j \leq N$ ,  $|x(t) - x_j(t)| > \varepsilon$  then  $dv(t)/dt = 0$ . Otherwise there exists a  $1 \leq j \leq N$  such that  $|x(t) - x_j(t)| = \varepsilon$  and the tagged particle experiences an instantaneous change of velocity. Define the collision parameter  $\nu \in \mathbb{S}^2$  by,

$$\nu := \frac{x(t) - x_j(t)}{|x(t) - x_j(t)|}.$$

Then the velocity of the tagged particle instantaneously after the collision,  $v(t)$ , is given by

$$v(t) := v(t^-) - \nu \cdot (v(t^-) - v_j)\nu.$$

It is noted that in this model we do not have conservation of momentum. The

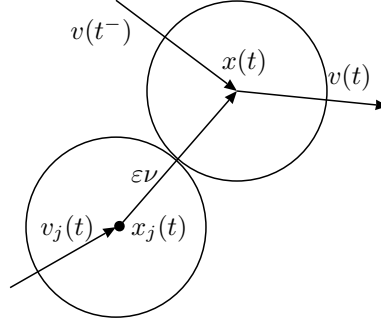


Figure 2-1: The collision parameter  $\nu$  within the dynamics of an example collision

background particles do not change velocity and the root particle does. We also note that these particle dynamics are irreversible because the background particles do not change velocity at a collision and so the incoming velocities of the tagged particle and the background particle cannot be recovered by the outgoing velocities.

**Proposition 2.2.1.** *For  $N \in \mathbb{N}$  and  $T > 0$  fixed these dynamics are well defined up to time  $T$  for all initial configurations apart from a set of measure zero.*

The proof is given in section 2.6, which establishes that almost surely all collisions involve only pairs and that there are only finitely many collisions in finite time.

We are interested in studying the distribution of a tagged particle among  $N$  background particles,  $\hat{f}_t^N$ , under the above particle dynamics as  $N$  tends to infinity or equivalently as  $\varepsilon$  tends to zero.

**Definition 2.2.2.** Probability densities  $f_0 \in L^1(U \times \mathbb{R}^3)$ ,  $g_0 \in L^1(\mathbb{R}^3)$  are *admissible* if

$$\int_{U \times \mathbb{R}^3} f_0(x, v)(1 + |v|^2) dx dv < \infty, \quad (2.2.2)$$

$$\int_{\mathbb{R}^3} g_0(v)(1 + |v|^2) dv < \infty, \quad (2.2.3)$$

$$\text{ess sup}_{v \in \mathbb{R}^3} g_0(v)(1 + |v|^5) < \infty. \quad (2.2.4)$$

**Remark 2.2.3.** Condition (2.2.4) can be relaxed to

$$\text{ess sup}_{v \in \mathbb{R}^3} g_0(v)(1 + |v|^{4+\eta}) < \infty \quad (2.2.5)$$

for some  $\eta > 0$ . Assumption (2.2.4) or (2.2.5) is required in section 2.3 in order to produce a bound needed to prove that honesty results for the semigroup defining the solution to the linear Boltzmann equation hold.

The distribution of the tagged particle is shown to converge to the solutions of a linear Boltzmann equation up to a finite arbitrary time  $T$ . We now state the main theorem of this chapter.

**Theorem 2.2.4.** *Let  $0 < T < \infty$  and  $f_0, g_0$  be admissible. Then  $\hat{f}_t^N$  converges to a time-dependent density  $f_t$  in the TV sense. Moreover, the limit  $f_t$  satisfies the linear Boltzmann equation*

$$\begin{cases} \partial_t f_t(x, v) &= -v \cdot \nabla_x f_t(x, v) + Q[f_t](x, v), \\ f_{t=0}(x, v) &= f_0(x, v), \end{cases} \quad (2.2.6)$$

where the collision operator  $Q$  is defined by  $Q := Q^+ - Q^-$  and  $Q^+$  and  $Q^-$  known respectively as the gain and loss term are given as follows,

$$Q^+[f](x, v) = \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} f(x, v') g_0(\bar{v}') [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu,$$

where the pre-collision velocities,  $v'$  and  $\bar{v}'$ , are given by  $v' = v + \nu \cdot (\bar{v} - v)\nu$  and  $\bar{v}' = \bar{v} - \nu \cdot (\bar{v} - v)\nu$ , and

$$Q^-[f](x, v) = f(x, v) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu.$$

### 2.2.1 Remarks

1. The reader is reminded that solutions of (2.2.6) only conserve mass, but not energy.
2. The analysis of the Rayleigh gas can also be done using traditional BBGKY approach. Here one uses the collision operator in  $\mathbb{R}^d$ :

$$\begin{aligned} \mathcal{C}_{s,s+1}^{\text{Rayl}} f^{(s+1)}(t, Z_s) &= (N - s) \varepsilon^{d-1} \int_{S^{d-1} \times \mathbb{R}^d} \nu \cdot (v_{s+1} - v_i) \\ &\quad \times f_N^{(s+1)}(t, Z_s, x_1 + \varepsilon \nu, v_{s+1}) d\nu dv_{s+1}. \end{aligned}$$

The hard sphere collision operator is given by

$$\begin{aligned} \mathcal{C}_{s,s+1}^{\text{hs}} f^{(s+1)}(t, Z_s) &:= (N - s) \varepsilon^{d-1} \sum_{i=1}^s \int_{S^{d-1} \times \mathbb{R}^d} \nu \cdot (v_{s+1} - v_i) \\ &\quad \times f_N^{(s+1)}(t, Z_s, x_i + \varepsilon \nu, v_{s+1}) d\nu dv_{s+1}. \end{aligned} \quad (2.2.7)$$

The only difference between  $\mathcal{C}_{s,s+1}^{\text{Rayl}}$  and  $\mathcal{C}_{s,s+1}^{\text{hs}}$  is the fact that in the hard sphere

case one sums over all indices  $i = 1, \dots, s$  and in the Rayleigh case only over  $i = 1$ . This gives estimates on  $\mathcal{C}_{s,s+1}^{\text{Rayl}}$ , which are independent of  $s$  in the contraction proof for the mild form of the associated BBGKY hierarchy. Using the function spaces  $\mathbf{X}_{\varepsilon,\beta,\mu}$  with norm  $\|\cdot\|_{\varepsilon,\beta,\mu}$  as in [20, Def 5.1.4] for measurable functions  $G : t \in [0, T] \mapsto (t) = (g_s(t))_{s \geq 1} \in \mathbf{X}_{\varepsilon,\beta,\mu}$  one can introduce another time-dependent variant compared to [20, Def 5.1.4]

$$\|G\|_{\varepsilon,\beta,\mu,\lambda} := \sup_{0 \leq t \leq T} \|G(t) \exp(-\lambda t |v_1|^2)\|_{\varepsilon,\beta,\mu}.$$

For a slightly different approach assuming only finite moments see [46, Section II.B].

3. Our method can be used to derive quantitative error estimates at the expense of more complex notation and additional regularity requirements for  $f_0$  and  $g_0$ . In particular, see lemmas 2.4.14 and 2.4.19 for some quantitative expressions.
4. The result should also hold in the case  $d = 2$  or  $d \geq 4$  up to a change in moment assumptions on the initial data and minor changes in estimates and calculations throughout.
5. A spatially inhomogeneous initial distribution for the background particles  $g_0 = g_0(x, v)$  is considered in chapter 3. This adds a complication to the equations since for example the operator  $Q$  in (2.2.6) becomes time-dependent, i.e.  $Q_t = Q_t^+ - Q_t^-$  with

$$Q_t^-[f](x, v) = f(x, v) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(x - t\bar{v}, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu,$$

and  $Q_t^+$  analogous. Since the operator now depends on the time  $t$  this requires evolution semigroup results to echo the semigroup results in [6].

6. One could also attempt to adapt these results to more complex and involved models. For example a model where each particle has an associated counter and a collision occurs between particle  $i$  and  $j$  if and only if both counters are less than  $k$ , in the hope of letting  $k$  tend to infinity. The main difficulty here will be that one will need to keep track of the current distribution of the background  $g_t$  in contrast to our model where the background has constant with time distribution  $g_0$ .

### 2.2.2 Method of Proof

We closely follow the method of [38]. That is we study the probability distribution of finding a given history of collisions at time  $t$ .

Firstly in section 2.3 we prove the main result of this section, theorem 2.3.1, which shows that there exists a solution  $P_t$  to a Kolmogorov differential equation and relate this solution to the solution of the linear Boltzmann equation. We show existence by explicitly building a solution on the most simple histories and using this to iteratively build a full solution.

In section 2.4 we consider the distribution  $\hat{P}_t$  of finding a given history of collisions from our particle dynamics and show by direct calculation that this solves a similar differential equation in theorem 2.4.6 for sufficiently well controlled (good) histories.

Finally in section 2.5 we prove the main theorem of this chapter, theorem 2.2.4, by proving the convergence between  $P_t$  and  $\hat{P}_t$  in theorem 2.5.8 and then relating this to  $f_t$  and  $\hat{f}_t^N$ .

### 2.2.3 Collision Histories

We construct trees in a similar way to [38]. Since the background particles only collide with the tagged particle and not each other we only consider trees of height at most 1 and so the trees simplify to the initial position of the tagged particle and a list of its collisions. Therefore the graph structure of the tree, used in [38], is not relevant here. Instead of tree in this thesis we use the terminology (collision) history or list. We note that if we were to attempt this method with a background that includes self interactions then we would need to use the tree structure as in [38].

A collision history includes the an initial position and velocity of the tagged particle  $(x_0, v_0) \in U \times \mathbb{R}^3$  along with a list of collisions that the tagged particle experiences. Each collision is denoted  $(t_j, \nu_j, v_j) \in (0, T] \times \mathbb{S}^2 \times \mathbb{R}^3$ , where  $t_j$  represents the collision time,  $\nu_j$  the collision parameter and  $v_j$  the incoming velocity of the background particle.

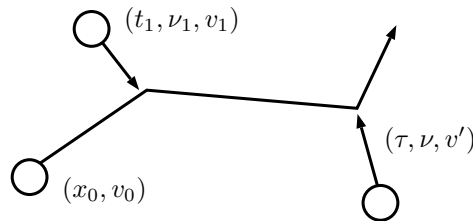


Figure 2-2: An example history

Consider the example collision history in figure 2-2: the tagged particle is initially at  $(x_0, v_0)$ . At time  $t_1$  it collides with a background particle with incoming velocity



$v_1$  and collision parameter  $\nu_1$ . Then after colliding and changing direction the tagged particle collides again with a background particle at time  $\tau$  with incoming velocity  $v'$  and collision parameter  $\nu$ . This example is represented by the collision history  $((x_0, v_0), (t_1, \nu_1, v_1), (\tau, \nu, v'))$

**Definition 2.2.5.** The set of collision histories  $\mathcal{MT}$  is defined by,

$$\mathcal{MT} := \{(x_0, v_0), (t_1, \nu_1, v_1), \dots, (t_n, \nu_n, v_n) : n \in \mathbb{N} \cup \{0\}, 0 < t_1 < \dots < t_n\}.$$

For a history  $\Phi \in \mathcal{MT}$  define  $n(\Phi)$  as the number of collisions.

The final collision in a history  $\Phi$  plays a significant role. Define the maximum collision time  $\tau(\Phi) \in [0, T]$ ,

$$\tau(\Phi) := \begin{cases} 0 & \text{if } n(\Phi) = 0, \\ t_n & \text{else.} \end{cases} \quad (2.2.8)$$

Further for  $n(\Phi) \geq 1$  for ease and to help denote the significance of the the marker for the final collision we introduce the notation,

$$(\tau, \nu, v') := (t_{n(\Phi)}, \nu_{n(\Phi)}, v_{n(\Phi)}).$$

The realisation of a history  $\Phi$  at a time  $t \in [0, T]$  for a particle diameter  $\varepsilon > 0$  uniquely defines the position and velocity of the tagged particle for all times up to  $t$  since the initial position and the collisions the tagged particle experiences are known. Further it determines the initial positions of the  $n(\Phi)$  background particles involved in the collision history since we can work backwards from the collision and we know that their velocity does not change. Finally it also includes information about the other  $N - n$  background particles, because it is known that they do not interfere with the tagged particle up to time  $t$ .

If the tagged particle collides at the instant  $t$  denote the pre-collisional velocity by  $v(t^-)$  and the post-collisional velocity by  $v(t)$ . Throughout this chapter the dependence on  $\Phi$  is often dropped from these and other variables when the context is clear.

Let  $\Phi \in \mathcal{MT}$ . For  $n \geq 1$ , define  $\bar{\Phi}$  as the collision history identical to  $\Phi$  but with the final collision removed. For example if  $\Phi = ((x_0, v_0), (t_1, \nu_1, v_1), (\tau, \nu, v'))$  then  $\bar{\Phi} = ((x_0, v_0), (t_1, \nu_1, v_1))$ .

Let  $\Phi \in \mathcal{MT}$ . Define  $\Phi_0 := (x_0, v_0)$  and for  $1 \leq j \leq n$  define  $\Phi_j = (t_j, \nu_j, v_j)$ . Define a metric  $d$  on  $\mathcal{MT}$  as follows. For  $\Phi, \Psi \in \mathcal{MT}$  with components  $\Phi_j$  and  $\Psi_j$

respectively.

$$d(\Phi, \Psi) := \begin{cases} 1 & \text{if } n(\Phi) \neq n(\Psi) \\ \min \{1, \max_{0 \leq j \leq n} |\Phi_j - \Psi_j|_\infty\} & \text{else.} \end{cases}$$

Further denote by  $B_h(\Phi)$  the ball of radius  $h/2$  around  $\Phi \in \mathcal{MT}$ ,

$$B_h(\Phi) := \{\Psi \in \mathcal{MT} : d(\Phi, \Psi) < h/2\}. \quad (2.2.9)$$

The standard Lebesgue measure on  $\mathcal{MT}$  is denoted by  $d\lambda$ .

## 2.3 The Idealised Distribution

In this section we show that there exists a solution, denoted  $P_t$ , to the idealised equation and relate this solution to the solution of the linear Boltzmann equation. We prove existence by constructing a solution iteratively on different sized collision histories. In section 2.5  $P_t$  is compared to the solution of a similar evolution equation defined by the particle dynamics in order to show the required convergence.

The idealised equation should be thought of as equivalent to the linear Boltzmann equation but written on the space  $\mathcal{MT}$  instead of on  $U \times \mathbb{R}^3$  and indeed by taking the appropriate marginal of the solution of the idealised equation we arrive at the solution of the linear Boltzmann equation. Within the idealised equation we consider the limiting dynamics of the particle system, where there are infinitely many particles acting as point masses.

Before we state the equation we first describe the structure of the equation with two examples. Consider first the history  $\Phi \in \mathcal{MT}$  with  $\Phi = ((x_0, v_0))$ , that is the tagged particle begins at  $(x_0, v_0)$  as has no collisions. Initially the probability of seeing this event is simply  $f_0(x_0, v_0)$ . Then for all  $t > 0$  the probability of seeing this event decays at a rate found by integrating over all possible collisions that the tagged particle can experience - since if the tagged particle experiences a collision then the event  $\Phi$  no longer exists.

Now consider instead the history  $\Phi = ((x_0, v_0), (t_1, \nu_1, v_1), (\tau, \nu, v'))$ . In this case initially the probability of seeing this event is zero, because it includes collisions that occur at time  $t_1$  and  $\tau$  and so the event cannot have occurred at the initial time. The probability of this event remains zero until  $t = \tau$ , at which there is an instantaneous jump to some positive probability. This probability is given by the probability of seeing  $\bar{\Phi} = ((x_0, v_0), (t_1, \nu_1, v_1))$  at time  $\tau$  and the probability that the final collision of  $\Phi$ ,  $(\tau, \nu, v')$  occurs. For all  $t > \tau$  again the probability of seeing  $\Phi$  decays at a rate

given by integrating over all possible collisions that the tagged particle can experience.

With that in mind the idealised equation is given by,

$$\begin{cases} \partial_t P_t(\Phi) &= \mathcal{Q}_t[P_t](\Phi) = \mathcal{Q}_t^+[P_t](\Phi) - \mathcal{Q}_t^-[P_t](\Phi), \\ P_0(\Phi) &= f_0(x_0, v_0) \mathbb{1}_{n(\Phi)=0}, \end{cases} \quad (2.3.1)$$

where,

$$\mathcal{Q}_t^+[P_t](\Phi) := \begin{cases} \delta(t - \tau) P_t(\bar{\Phi}) g_0(v') [(v(\tau^-) - v') \cdot \nu]_+ & \text{if } n \geq 1 \\ 0 & \text{if } n = 0. \end{cases} \quad (2.3.2)$$

$$\mathcal{Q}_t^-[P_t](\Phi) := P_t(\Phi) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu. \quad (2.3.3)$$

**Theorem 2.3.1.** *Suppose that  $f_0$  and  $g_0$  are admissible (in the sense of Def. 2.2.2). Then there exists a solution  $P : [0, T] \rightarrow L^1(\mathcal{MT})$  to the idealised equation, (2.3.1). Moreover for any  $t \in [0, T]$  and for any  $\Omega \subset U \times \mathbb{R}^3$  define*

$$S_t(\Omega) := \{\Phi \in \mathcal{MT} : (x(t), v(t)) \in \Omega\}.$$

Then

$$\int_{\Omega} f_t(x, v) dx dv = \int_{S_t(\Omega)} P_t(\Phi) d\Phi,$$

where  $f_t$  is the unique mild solution of the linear Boltzmann equation given in proposition 2.3.4.

From now on assume that  $f_0$  and  $g_0$  are admissible with the provision that either (2.2.4) or (2.2.5) holds. We prove the existence by construction, taking several steps to build a solution by solving on the most simple histories first and using this solution to iteratively build a full solution. We begin by solving the linear Boltzmann equation. We establish existence, uniqueness and regularity of solutions of (2.2.6) by adapting methods from semigroup theory. The difficulty here is that after writing the linear Boltzmann equation as the sum of two unbounded operators we need to ensure that a honest semigroup is generated in order to prove existence and uniqueness. Next we adapt these semigroup techniques to define functions  $P_t^{(j)}$  that describe the distribution of finding the tagged particle such that it has experienced  $j$  collisions. This is key to connecting  $P_t$  to the solution of the linear Boltzmann.

The following notion of mild solution suitable for transport equations is used (c.f. [2, Def 3.1.1])

**Definition 2.3.2.** Consider the following system,

$$\begin{cases} \partial_t u(t) &= Lu(t), \\ u(0) &= u_0. \end{cases} \quad (2.3.4)$$

Where  $L : D(L) \subset L^1(U \times \mathbb{R}^3) \rightarrow L^1(U \times \mathbb{R}^3)$  is an operator and  $u_0 \in L^1(U \times \mathbb{R}^3)$  is given. The function  $u : [0, T] \rightarrow U \times \mathbb{R}^3$  is called a *mild solution* of (2.3.4) if for all  $t \geq 0$ ,

$$\int_0^t u(\theta) d\theta \in D(L) \text{ and } L \int_0^t u(\theta) d\theta = u(t) - u_0.$$

We split the right hand side of (2.2.6) into two operators,  $A$  and  $B$ . These will appear in the construction of  $P_t$ .

**Definition 2.3.3.** Define  $D(A), D(B) \subset L^1(U \times \mathbb{R}^3)$  by,

$$\begin{aligned} D(A) &:= \{f \in L^1(U \times \mathbb{R}^3) : v \cdot \nabla_x f(x, v) + Q^-[f](x, v) \in L^1(U \times \mathbb{R}^3)\}, \\ D(B) &:= \{f \in L^1(U \times \mathbb{R}^3) : Q^+[f] \in L^1(U \times \mathbb{R}^3)\}. \end{aligned}$$

Then define  $A : D(A) \rightarrow L^1(U \times \mathbb{R}^3)$  and  $B : D(B) \rightarrow L^1(U \times \mathbb{R}^3)$  by,

$$(Af)(x, v) := -v \cdot \nabla_x f(x, v) - Q^-[f](x, v), \quad (2.3.5)$$

$$(Bf)(x, v) := Q^+[f](x, v). \quad (2.3.6)$$

**Proposition 2.3.4.** Suppose that the assumptions in theorem 2.3.1 hold. Then there exists a unique mild solution  $f : [0, T] \rightarrow L^1(U \times \mathbb{R}^3)$  to (2.2.6). Furthermore  $f_t$  remains non-negative and of mass 1, and

$$\int_{U \times \mathbb{R}^3} f_t(x, v)(1 + |v|) dx dv < \infty, \quad (2.3.7)$$

$$f_t \in D(B) \quad (2.3.8)$$

hold for all  $t \in [0, T]$ .

To prove this proposition we employ semigroup techniques which first requires two lemmas.

**Lemma 2.3.5.**  $A$  is the generator of the substochastic semigroup (c.f. [6, Section 10.2])  $T(t) : L^1(U \times \mathbb{R}^3) \rightarrow L^1(U \times \mathbb{R}^3)$  given by,

$$(T(t)f)(x, v) := \exp \left( -t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v})[(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \right) f(x - tv, v). \quad (2.3.9)$$

*Proof.* We seek to apply theorem 10.4 of [6]. Conditions (A1), (A2) trivially hold since  $F = 0$  in our situation. As for (A3),

$$\nu(x, v) = \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu. \quad (2.3.10)$$

This is locally integrable so (A3) holds. Hence we can apply the theorem. In our case,

$$\varphi(x, v, t, s) = x - (t + s)v.$$

So the semigroup is given by

$$(T(t)f)(x, v) := \exp \left( -t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \right) f(x - tv, v),$$

as required.  $\square$

**Remark 2.3.6.** For  $v, v_* \in \mathbb{R}^3, v \neq v_*$  define the Boltzmann kernel  $k$  by,

$$k(v, v_*) := \frac{1}{|v - v_*|} \int_{E_{v, v_*}} g_0(w) dw, \quad (2.3.11)$$

where  $E_{v, v_*} = \{w \in \mathbb{R}^3 : w \cdot (v - v_*) = v \cdot (v - v_*)\}$ . By the use of Carleman's representation, first described in [12] (see also [8, Section 3]),

$$\begin{aligned} (Bf)(x, v) &= Q^+[f](x, v) \\ &= \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} f(x, v') g_0(\bar{v}') [(v - \bar{v}') \cdot \nu]_+ d\bar{v}' d\nu \\ &= \int_{\mathbb{R}^3} k(v, v_*) f(x, v_*) dv_*. \end{aligned} \quad (2.3.12)$$

**Lemma 2.3.7.** *There exists a  $C > 0$  such that for any  $V > 0$ ,*

$$\int_{|v| > V} k(v, v_*) dv \leq C, \quad (2.3.13)$$

*for almost all  $|v_*| \leq V$ .*

*Proof.* Define  $\bar{g} : [0, \infty) \rightarrow \mathbb{R}$  by

$$\bar{g}(z) = \sup_{\{|v|^2=z\}} g_0(v),$$

and define  $h : [0, \infty) \rightarrow \mathbb{R}$  by

$$h(r) := \int_0^\infty \sigma \sup_{|v|^2 = \sigma^2 + r} g_0(v) \, d\sigma = \int_0^\infty \sigma \bar{g}(\sigma^2 + r) \, d\sigma.$$

Then

$$g_0(v) \leq \sup_{\{|u|^2 = |v|^2\}} g_0(u) = \bar{g}(|v|^2).$$

Now define

$$r := \frac{1}{2} \left( |v - v_*| + \frac{|v|^2 - |v_*|^2}{|v - v_*|} \right).$$

Following the calculations and notation in the proof of [3, Theorem 2.1] we have

$$\begin{aligned} \int_{E_{vv_*}} g_0(w) \, dw &= \int_{V_2 \cdot (v - v_*) = 0} g_0(V_2 + v) \, dV_2 \\ &\leq \int_{V_2 \cdot (v - v_*) = 0} \bar{g}(|V_2 + v|^2) \, dV_2 \\ &= \int_{V_2 \cdot (v - v_*) = 0} \bar{g}(|u_\perp + V_2|^2 + r^2) \, dV_2 \\ &= \int_{\mathbb{R}^2} \bar{g}(|\bar{v}|^2 + r^2) \, d\bar{v} \\ &= 2\pi \int_0^\infty \sigma \bar{g}(\sigma^2 + r^2) \, d\sigma \\ &= 2\pi h(r^2). \end{aligned}$$

Hence,

$$k(v, v_*) \leq \frac{2\pi}{|v - v_*|} h(r^2).$$

Thus we can follow the calculations in [6, Ex 10.29] to see that, for  $d = 3$ , for almost all  $|v^*| \leq V$ ,

$$\int_{|v| > V} k(v, v_*) \, dv \leq C\Gamma,$$

where  $C > 0$  is some geometric constant and

$$\Gamma := \int_0^\infty r h(r^2) \, dr.$$

Assuming either (2.2.4) or (2.2.5)  $\Gamma$  is finite and the result is proved.  $\square$

*Proof of proposition 2.3.4.* By the definition of  $A$  and  $B$ , (2.2.6) is now,

$$\begin{cases} \partial_t f_t(x, v) &= (Af + Bf)(x, v) \\ f_{t=0}(x, v) &= f_0(x, v). \end{cases} \quad (2.3.14)$$

By lemma 2.3.5 and equations (10.86), (10.88) of [6], [6, corollary 5.17] holds for our equation, so we apply [6, theorem 5.2]. Lemma 2.3.7 allow us to further apply [6, theorem 10.28]. Therefore we have an honest  $C_0$  semigroup of contractions generated by  $\overline{A + B}$ , which we denote by  $G(t)$ . Finally by [2, theorem 3.1.12], (2.3.14) has a unique mild solution for each  $f_0 \in L^1(U \times \mathbb{R}^3)$ .

It remains to show (2.3.7) and (2.3.8). Firstly we find a bound for the operator  $B$ . Then we prove (2.3.7) for  $f_0 \in D(A)$  first before generalising to all  $f_0$ . Recall (2.4.17). As  $g_0$  is normalized,

$$\begin{aligned} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v})[(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu &\leq \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v})|v - \bar{v}| d\bar{v} d\nu \\ &\leq \pi \int_{\mathbb{R}^3} g_0(\bar{v})(|v| + |\bar{v}|) d\bar{v} d\nu \leq \pi(|v| + \beta). \end{aligned} \quad (2.3.15)$$

By (10.6) in [6] for  $f \in L^1(U \times \mathbb{R}^3)$  and by (2.3.15) recalling (2.3.10),

$$\begin{aligned} \int_{U \times \mathbb{R}^3} Bf(x, v) dx dv &= \int_{U \times \mathbb{R}^3} \int_{\mathbb{R}^3} k(v, v_*) f(x, v_*) dv_* dx dv \\ &= \int_{U \times \mathbb{R}^3} \nu(x, v) f(x, v) dx dv \\ &= \int_{U \times \mathbb{R}^3} f(x, v) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v})[(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu dx dv \\ &\leq \int_{U \times \mathbb{R}^3} f(x, v) \pi(|v| + \beta) dx dv. \end{aligned} \quad (2.3.16)$$

Now suppose that  $f_0 \in D(A)$ . Then [6, corollary 5.17] holds and so we apply [6, corollary 5.8] which gives,

$$f_t = T(t)f_0 + \int_0^t G(t - \theta)BT(\theta)f_0 d\theta. \quad (2.3.17)$$

Where  $G(t)$  is the contraction semigroup generated by  $\overline{A + B}$  and  $T(t)$  the contraction

semigroup generated by  $A$ . Hence by (2.3.16),

$$\begin{aligned} \int_{U \times \mathbb{R}^3} f_t(x, v)(1 + |v|) \, dx \, dv & \\ & \leq \int_{U \times \mathbb{R}^3} T(t)f_0(x, v)(1 + |v|) + \int_0^t G(t - \theta)BT(\theta)f_0 \, d\theta(x, v)(1 + |v|) \, dx \, dv \\ & \leq \int_{U \times \mathbb{R}^3} f_0(x, v)(1 + |v|) + tf_0(x, v)(1 + |v|)\pi(|v| + \beta) \, dx \, dv. \end{aligned}$$

Noting that  $t \in [0, T]$  and recalling our assumption on  $f_0$  in (2.2.2), this is bounded.

Consider a general  $f_0$  in (2.2.2), not necessarily in  $D(A)$ . Suppose for contradiction that (2.3.7) is not true. Hence there exists a  $t \in [0, T]$  such that for any  $C > 0$  there exists an  $R > 0$  such that,

$$\int_U \int_{B_R(0)} f_t(x, v)(1 + |v|) \, dv \, dx \geq C.$$

$D(A)$  is dense in  $L^1(U \times \mathbb{R}^3)$  because it contains, for example, smooth compactly supported functions. Hence for any  $n \in \mathbb{N}$  there exists an  $f_0^n \in L^1(U \times \mathbb{R}^3)$  such that  $f_0^n \in D(A)$ ,  $f_0^n \geq 0$ , there exists a  $C_1 > 0$  such that

$$\int_{U \times \mathbb{R}^3} f_0^n(x, v)(1 + |v|^2) \, dx \, dv \leq C_1, \quad (2.3.18)$$

and that,

$$\int_{U \times \mathbb{R}^3} |f_0^n(x, v) - f_0(x, v)|(1 + |v|) \, dx \, dv \leq \frac{1}{n}. \quad (2.3.19)$$

Now define  $f_t^n$  to be the solution of (2.2.6) with initial data given by  $f_0^n$ . In this case (2.3.17) gives,

$$f_t^n = T(t)f_0^n + \int_0^t G(t - \theta)BT(\theta)f_0^n \, d\theta.$$

The argument above for  $f_0^n \in D(A)$  together with (2.3.18) gives that there exists a  $C_2 > 0$  independent of  $n$  such that,

$$\int_{U \times \mathbb{R}^3} f_t^n(x, v)(1 + |v|) \, dx \, dv \leq C_2.$$

By our contradiction assumption there exists an  $R > 0$  such that,

$$\int_U \int_{B_R(0)} f_t(x, v)(1 + |v|) \, dv \, dx \geq 2C_2.$$



These two bounds together give that,

$$\begin{aligned}
& \int_{U \times \mathbb{R}^3} |f_t(x, v) - f_t^n(x, v)|(1 + |v|) \, dx \, dv \\
& \geq \int_U \int_{B_R(0)} |f_t(x, v) - f_t^n(x, v)|(1 + |v|) \, dv \, dx \\
& \geq \left| \int_U \int_{B_R(0)} f_t(x, v)(1 + |v|) \, dv \, dx - \int_U \int_{B_R(0)} f_t^n(x, v)(1 + |v|) \, dv \, dx \right| \\
& \geq C_2.
\end{aligned} \tag{2.3.20}$$

However since  $G(t)$  is a contraction semigroup it follows for  $n > 1/C_2$  by (2.3.19),

$$\begin{aligned}
& \int_{U \times \mathbb{R}^3} (f_t(x, v) - f_t^n(x, v))(1 + |v|) \, dx \, dv \\
& = \int_{U \times \mathbb{R}^3} G(t)(f_0 - f_0^n)(x, v)(1 + |v|) \, dx \, dv \\
& \leq \int_{U \times \mathbb{R}^3} (f_0(x, v) - f_0^n(x, v))(1 + |v|) \, dx \, dv \leq \frac{1}{n} < C_2.
\end{aligned}$$

By the same argument,

$$\int_{U \times \mathbb{R}^3} (f_t^n(x, v) - f_t(x, v))(1 + |v|) \, dx \, dv < C_2.$$

Hence we have a contradiction with (2.3.20) which completes the proof of (2.3.7).

To show (2.3.8) fix  $t \in [0, T]$ . By (2.3.16),

$$\int_{U \times \mathbb{R}^3} B f_t(x, v) \, dx \, dv \leq \int_{U \times \mathbb{R}^3} f_t(x, v) \pi(|v| + \beta) \, dx \, dv.$$

By the above calculations  $f_t$  has finite first moment so this is finite as required.  $\square$

**Proposition 2.3.8.** *There exists a unique mild solution,  $P^{(0)} : [0, T] \rightarrow L^1(U \times \mathbb{R}^3)$ , to the following evolution equation,*

$$\begin{cases} \partial_t P_t^{(0)}(x, v) &= (A P_t^{(0)})(x, v), \\ P_0^{(0)}(x, v) &= f_0(x, v). \end{cases} \tag{2.3.21}$$

Where  $A$  is as in (2.3.5).

The distribution  $P_t^{(0)}(x, v)$  can be thought of as the probability of finding the tagged particle at  $(x, v)$  at time  $t$  such that it has not yet experienced any collisions.

*Proof.* By lemma 2.3.5  $A$  generates the substochastic  $C_0$  semigroup  $T(t)$  given in

(2.3.9). By the Hille-Yoshida theorem, [40, Thm 1.3.1]  $A$  is closed. By [2, Thm 3.1.12] (2.3.21) has a unique mild solution given by

$$P_t^{(0)} = T(t)f_0. \quad (2.3.22)$$

□

**Lemma 2.3.9.** *For all  $t \in [0, T]$ ,  $P_t^{(0)} \leq f_t$  pointwise.*

**Remark 2.3.10.** This lemma is entirely expected. The probability of finding the tagged particle at  $(x, v)$  at time  $t$  is given by  $f_t(x, v)$  and the probability of finding it at  $(x, v)$  at time  $t$  such that it has not experienced any collisions up to time  $t$  is given by  $P_t^{(0)}(x, v)$  so one expects  $P_t^{(0)} \leq f_t$ .

*Proof.* For  $t \in [0, T]$  define  $F_t^{(0)} := f_t - P_t^{(0)}$ . Then since  $f_t$  and  $P_t^{(0)}$  are mild solutions of (2.2.6) and (2.3.21) respectively,  $F_t^{(0)}$  is a mild solution of

$$\begin{cases} \partial_t F_t^{(0)}(x, v) &= A F_t^{(0)} + B f_t(x, v) \\ F_0^{(0)}(x, v) &= 0. \end{cases}$$

By (2.3.8) and [2, Prop. 3.1.16]  $F_t^{(0)}$  is given by,

$$F_t^{(0)} = \int_0^t T(t - \theta) B f_\theta d\theta.$$

Now noting that  $f_\theta$  is non-negative it follows that  $B f_\theta$  and hence  $T(t - \theta) B f_\theta$  are non-negative also. Hence  $F_t^{(0)} \geq 0$  which implies  $P_t^{(0)} \leq f_t$ . □

**Definition 2.3.11.** For  $j \in \mathbb{N} \cup \{0\}$  denote by  $\mathcal{T}_j$  the set of all collision histories with exactly  $j$  collisions. Explicitly,

$$\mathcal{T}_j := \{\Phi \in \mathcal{MT} : n(\Phi) = j\}. \quad (2.3.23)$$

The required solution  $P_t$  can now be defined iteratively on the space  $\mathcal{MT}$ . For  $\Phi \in \mathcal{T}_0$  define

$$P_t(\Phi) := P_t^{(0)}(x(t), v(t)). \quad (2.3.24)$$

Else define,

$$\begin{aligned} P_t(\Phi) := \mathbb{1}_{t \geq \tau} \exp \left( -(t - \tau) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(\tau) - \bar{v}) \cdot \nu']_+ d\bar{v} d\nu' \right) \\ P_\tau(\bar{\Phi}) g_0(v') [(v(\tau^-) - v') \cdot \nu]_+. \end{aligned} \quad (2.3.25)$$

The right hand side of this equation depends on  $P_\tau(\bar{\Phi})$  but since  $\bar{\Phi}$  has exactly one less collision than  $\Phi$  the equation is well defined.

The proof that  $P_t$  has the required properties of theorem 2.3.1 is given shortly. We first define the function  $P_t^{(j)}$  which is thought of, in parallel to  $P_t^{(0)}$ , as being the probability of finding the tagged particle at a certain position at time  $t$  such that it has experienced exactly  $j$  collisions up to time  $t$ . The  $P_t^{(j)}$  will be required to show the connection between  $P_t$  and the solution of the linear Boltzmann equation.

**Definition 2.3.12.** Let  $t \in [0, T]$  and  $\Omega \subset U \times \mathbb{R}^3$  be measurable. Recall in theorem 2.3.1 we define the set,  $S_t(\Omega) = \{\Phi \in \mathcal{MT} : (x(t), v(t)) \in \Omega\}$  - the set of all collision histories such that the tagged particle at time  $t$  is in  $\Omega$ . Define for all  $j \in \mathbb{N} \cup \{0\}$ ,

$$S_t^j(\Omega) := \mathcal{T}_j \cap S_t(\Omega).$$

Then for  $j \geq 1$ , define  $P^{(j)}(\Omega)$  by,

$$P_t^{(j)}(\Omega) := \int_{S_t^j(\Omega)} P_t(\Phi) d\Phi.$$

**Lemma 2.3.13.** Let  $t \in [0, T]$ ,  $j \geq 1$ . Then  $P_t^{(j)}$  is absolutely continuous with respect to the Lebesgue measure on  $U \times \mathbb{R}^3$ .

*Proof.* Let  $j = 1$ . Then we have by (2.3.25),

$$\begin{aligned} P_t^{(1)}(\Omega) &= \int_{S_t^1(\Omega)} P_t(\Phi) d\Phi \\ &= \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_U \exp \left( -(t - \tau) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \right) \\ &\quad P_\tau^{(0)}(x_0 + \tau v_0, v_0) g_0(v') [(v_0 - v') \cdot \nu]_+ \mathbb{1}_{(x(t), v(t)) \in \Omega} dx_0 d\nu dv' dv_0 d\tau \end{aligned} \quad (2.3.26)$$

We define a coordinate transform  $(\nu, x_0, v_0, v') \mapsto (\nu, x, v, \bar{w})$  given by,

$$\begin{aligned} v &:= v_0 + \nu(v' - v_0) \cdot \nu \\ x &:= x_0 + \tau v_0 + (t - \tau)v \\ \bar{w} &:= v' - \nu(v' - v_0) \cdot \nu. \end{aligned}$$

This transformation has the Jacobi matrix,

$$\begin{pmatrix} \text{Id} & 0 & 0 & 0 \\ & \text{Id} & & \\ & 0 & \text{Id} - \nu \otimes \nu & \nu \otimes \nu \\ & 0 & \nu \otimes \nu & \text{Id} - \nu \otimes \nu \end{pmatrix}$$

where the blank entries are not required for the computation of the matrix's determinant. The 2x2 matrix in the bottom right has determinant  $-1$  and hence the absolute value of the determinant of the entire matrix is 1. With this (2.3.26) becomes,

$$P_t^{(1)}(\Omega) = \int_{\Omega} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \exp \left( -(t - \tau) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \right) \\ P_{\tau}^{(0)}(x - (t - \tau)v, w') g_0(\bar{w}') [(v - \bar{w}') \cdot \nu]_+ d\nu d\bar{w} d\tau dx dv,$$

where  $w' = v + \nu(\bar{w} - v) \cdot \nu$  and  $\bar{w}' = \bar{w} - \nu(\bar{w} - v) \cdot \nu$ . Hence we see that if the Lebesgue measure of  $\Omega$  equals zero then so does  $P_t^{(1)}(\Omega)$ . For  $j \geq 2$  we use a similar approach, using the iterative formula for  $P_t(\Phi)$  (2.3.25).  $\square$

**Remark 2.3.14.** Since  $P_t^{(j)}$  is an absolutely continuous measure on  $U \times \mathbb{R}^3$ , the Radon-Nikodym theorem (see Theorem 4.2.2 [15]) implies that  $P_t^{(j)}$  has a density, which we denote by  $P_t^{(j)}$  also. This gives,

$$\int_{\Omega} P_t^{(j)}(x, v) dx dv = \int_{S_t^j(\Omega)} P_t(\Phi) d\Phi.$$

Hence for almost all  $(x, v) \in U \times \mathbb{R}^3$ ,

$$P_t^{(j)}(x, v) = \int_{S_t^j(x, v)} P_t(\Phi) d\Phi.$$

**Remark 2.3.15.** A similar formula holds for  $P_t^{(0)}$  since the set  $S_t^0(x, v)$  contains exactly one collision history:  $\Phi = ((x - tv, v))$ ,

$$\int_{S_t^0(x, v)} P_t(\Phi) d\Phi = P_t((x - tv, v)) = P_t^{(0)}(x, v).$$

**Proposition 2.3.16.** For  $j \geq 1$ ,  $P_t^{(j)}$  as defined above is almost everywhere the unique

mild solution to the following differential equation,

$$\begin{cases} \partial_t P_t^{(j)}(x, v) &= AP_t^{(j)}(x, v) + BP_t^{(j-1)}(x, v), \\ P_0^{(j)}(x, v) &= 0, \end{cases}$$

where  $A$  is given in (2.3.5) and  $B$  in (2.3.6).

The following lemma helps prove the proposition for the case  $j = 1$  which allows the use of an inductive argument to prove the proposition in full.

**Lemma 2.3.17.** *For any  $t \in [0, T]$  and almost all  $(x, v) \in U \times \mathbb{R}^3$ ,*

$$P_t^{(1)}(x, v) = \int_0^t T(t - \theta) BP_\theta^{(0)}(x, v) d\theta,$$

where the semigroup  $T(t)$  is as in (2.3.9). The right hand side is well defined since (2.3.8), (2.3.9) and (2.3.16) imply  $P_t^{(0)} \in D(B)$ .

*Proof.* We show that for any  $\Omega \subset U \times \mathbb{R}^3$  measurable we have,

$$\int_\Omega P_t^{(1)}(x, v) dx dv = \int_\Omega \int_0^t T(t - \theta) BP_\theta^{(0)}(x, v) d\theta dx dv.$$

By the definition of  $T(t)$  in equation (2.3.9), the definition of  $B$  in (2.3.6), and the proof of lemma 2.3.13, for  $w' = v + \nu(\bar{w} - v) \cdot \nu$  and  $\bar{w}' = \bar{w} - \nu(\bar{w} - v) \cdot \nu$ , we have,

$$\begin{aligned} & \int_\Omega \int_0^t T(t - \theta) BP_\theta^{(0)}(x, v) d\theta dx dv \\ &= \int_\Omega \int_0^t \exp \left( -(t - \theta) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \right) \\ & \quad \times BP_\theta^{(0)}(x - (t - \theta)v, v) d\theta dx dv \\ &= \int_\Omega \int_0^t \exp \left( -(t - \theta) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \right) \\ & \quad \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} P_\theta^{(0)}(x - (t - \theta)v, w') g_0(\bar{w}') [(v - \bar{w}') \cdot \nu]_+ d\nu d\bar{w}' d\theta dx dv \\ &= \int_\Omega \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \exp \left( -(t - \theta) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \right) \\ & \quad P_\theta^{(0)}(x - (t - \theta)v, w') g_0(\bar{w}') [(v - \bar{w}') \cdot \nu]_+ d\nu d\bar{w}' d\theta dx dv \\ &= \int_\Omega P_t^{(1)}(x, v) dx dv. \end{aligned}$$

□

*Proof of proposition 2.3.16.* Consider induction on  $j$ . First let  $j = 1$ . We seek to apply [2, Prop 3.1.16]. If  $\int_0^t BP_\theta^{(0)} d\theta \in L^1(U \times \mathbb{R}^3)$  then the proposition holds so by the above lemma  $P_t^{(1)}$  is the unique mild solution.

To this aim note that since  $P_t^{(0)}$  is the unique mild solution of (2.3.21),

$$\int_0^t P_\theta^{(0)} d\theta \in D(A).$$

By [6, Section 10.4.3]  $D(A) \subset D(B)$  and hence

$$\int_0^t P_\theta^{(0)} d\theta \in D(B).$$

This implies,

$$B \int_0^t P_\theta^{(0)} d\theta \in L^1(U \times \mathbb{R}^3)$$

as required. Now consider  $j \geq 2$  and assume the proposition is true for  $j-1$ . By setting  $F_t^{(j-1)} := f_t - P_t^{(j-1)}$  a similar argument to lemma 2.3.9 shows that  $P_t^{(j-1)} \leq f_t$ . By (2.3.8) and (2.3.16),  $P_t^{(j-1)} \in D(B)$  so the right hand side is well defined. A similar approach to lemma 2.3.17 shows that for any  $t \in [0, T]$  and almost all  $(x, v) \in U \times \mathbb{R}^3$ ,

$$P_t^{(j)}(x, v) = \int_0^t (T(t-\theta)BP_\theta^{(j-1)})(x, v) d\theta,$$

where  $T(t)$  is the semigroup given in (2.3.9). The rest follows by the same argument as in the  $j = 1$  case.  $\square$

**Proposition 2.3.18.** *For all  $t \in [0, T]$  and almost all  $(x, v) \in U \times \mathbb{R}^3$ ,*

$$\sum_{j=0}^{\infty} P_t^{(j)}(x, v) = f_t(x, v), \quad (2.3.27)$$

where  $f_t$  is the unique mild solution of the linear Boltzmann equation given in proposition 2.3.4.

*Proof.* Since  $P_t^{(0)}$  is a mild solution of (2.3.21),

$$\int_0^t P_\theta^{(0)}(x, v) d\theta \in D(A), \quad (2.3.28)$$

and,

$$P_t^{(0)}(x, v) = f_0(x, v) + A \int_0^t P_\theta^{(0)}(x, v) d\theta. \quad (2.3.29)$$

Further for  $j \geq 1$  by proposition 2.3.16 and [6, Prop 3.31],

$$\int_0^t P_\theta^{(j)}(x, v) d\theta \in D(A), \quad (2.3.30)$$

and,

$$P_t^{(j)}(x, v) = A \int_0^t P_\theta^{(j)}(x, v) d\theta + \int_0^t B P_\theta^{(j-1)}(x, v) d\theta. \quad (2.3.31)$$

Combining (2.3.28) and (2.3.30),

$$\int_0^t \sum_{j=0}^{\infty} P_\theta^{(j)}(x, v) d\theta \in D(A).$$

Recalling from the proof of proposition 2.3.16 that  $D(A) \subset D(B)$ ,

$$\int_0^t \sum_{j=0}^{\infty} P_\theta^{(j)}(x, v) d\theta \in D(A) \cap D(B) = D(A + B).$$

Further summing (2.3.29) and (2.3.31) for  $j \geq 1$ ,

$$\begin{aligned} \sum_{j=0}^{\infty} P_t^{(j)}(x, v) &= f_0(x, v) + \sum_{j=0}^{\infty} A \int_0^t P_\theta^{(j)}(x, v) d\theta + \sum_{j=1}^{\infty} \int_0^t B P_\theta^{(j-1)}(x, v) d\theta \\ &= f_0(x, v) + A \int_0^t \sum_{j=0}^{\infty} P_\theta^{(j)}(x, v) d\theta + \int_0^t B \sum_{j=0}^{\infty} P_\theta^{(j)}(x, v) d\theta \\ &= f_0(x, v) + (A + B) \int_0^t \sum_{j=0}^{\infty} P_\theta^{(j)}(x, v) d\theta. \end{aligned}$$

Hence by definition 2.3.2,  $\sum_{j=0}^{\infty} P_t^{(j)}(x, v)$  is a mild solution of (2.3.14) and therefore since  $f_t$  is the unique mild solution the proof is complete.  $\square$

We now have all the results needed to prove that  $P_t$  satisfies all the requirements of theorem 2.3.1.

*Proof of theorem 2.3.1.* Using definition 2.3.12, proposition 2.3.18, and, since each  $P_t^{(j)}$  is positive, the monotone convergence theorem we have for any measurable  $\Omega \subset U \times \mathbb{R}^3$ ,

$$\int_{S_t(\Omega)} P_t(\Phi) d\Phi = \sum_{j=0}^{\infty} \int_{S_t^j(\Omega)} P_t(\Phi) d\Phi = \sum_{j=0}^{\infty} \int_{\Omega} P_t^{(j)}(x, v) dx dv = \int_{\Omega} f_t(x, v) dx dv. \quad (2.3.32)$$

In particular,

$$\int_{\mathcal{MT}} P_t(\Phi) d\Phi = \int_{U \times \mathbb{R}^3} f_t(x, v) dx dv < \infty.$$

Hence  $P_t \in L^1(\mathcal{MT})$ . To show that  $P_t$  is a solution of (2.3.1) first consider  $\Phi \in \mathcal{T}_0$ . Since  $n(\Phi) = 0$ ,

$$P_0(\Phi) = P_0^{(0)}(x(0), v(0)) = f_0(x_0, v_0) = f_0(x_0, v_0) \mathbb{1}_{n(\Phi)=0}.$$

Hence it solves the initial condition. Now for  $t > 0$ , since  $\Phi \in \mathcal{T}_0$ ,  $v(t) = v_0$  and  $x(t) = x_0 + tv_0$ . Hence by (2.3.9) and (2.3.22),

$$\begin{aligned} P_t(\Phi) &= P_t^{(0)}(x(t), v(t)) = P_t^{(0)}(x_0 + tv_0, v_0) \\ &= \exp \left( -t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v_0 - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \right) f_0(x_0, v_0). \end{aligned}$$

The only dependence on  $t$  here is in the exponential term so we differentiate  $P_t(\Phi)$  with respect to  $t$ ,

$$\begin{aligned} \partial_t P_t(\Phi) &= \partial_t \left( \exp \left( -t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v_0 - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \right) f_0(x_0, v_0) \right) \\ &= - \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v_0 - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \times P_t(\Phi) = -\mathcal{Q}_t^- [P_t](\Phi). \end{aligned}$$

Hence  $P_t$  solves (2.3.1) on  $\mathcal{T}_0$ .

We now consider  $\Phi \in \mathcal{T}_j$  for  $j \geq 1$ . Since  $\Phi \in \mathcal{T}_j$  we have  $n(\Phi) = j$  and  $\tau > 0$ . Hence

$$P_0(\Phi) = 0 = f_0(x_0, v_0) \mathbb{1}_{n(\Phi)=0}. \quad (2.3.33)$$

For  $t = \tau$ ,

$$P_\tau(\Phi) = P_\tau(\bar{\Phi}) g_0(v') [v(\tau^-) - v'] \cdot \nu]_+. \quad (2.3.34)$$

Further for  $t > \tau$  the only dependence on  $t$  is inside the exponential term and hence



differentiating gives,

$$\begin{aligned}
\partial_t P_t(\Phi) &= \partial_t \left( \exp \left( - (t - \tau) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(\tau) - \bar{v}) \cdot \nu']_+ d\bar{v} d\nu' \right) \right. \\
&\quad \left. P_\tau(\bar{\Phi}) g_0(v') [(v_0 - v') \cdot \nu]_+ \right) \\
&= \exp \left( - (t - \tau) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(\tau) - \bar{v}) \cdot \nu']_+ d\bar{v} d\nu' \right) \\
&\quad P_\tau(\bar{\Phi}) g_0(v') [(v_0 - v') \cdot \nu]_+ \\
&\quad \times \left( - \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(\tau) - \bar{v}) \cdot \nu']_+ d\bar{v} d\nu' \right) \\
&= -P_t(\Phi) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(\tau) - \bar{v}) \cdot \nu']_+ d\bar{v} d\nu' = -\mathcal{Q}_t^-[P_t](\Phi). \tag{2.3.35}
\end{aligned}$$

Equations (2.3.33), (2.3.34) and (2.3.35) prove that  $P_t$  solves (2.3.1) on  $\mathcal{T}_j$ . Since  $\mathcal{MT}$  is the disjoint union of  $\mathcal{T}_j$  for  $j \geq 0$ ,  $P_t$  is a solution of (2.3.1) on  $\mathcal{MT}$ . Finally, the required connection between  $P_t$  and the solution of the linear Boltzmann equation has been shown in (2.3.32).  $\square$

## 2.4 The Empirical Distribution

We now consider the empirical distribution on collision histories  $\hat{P}_t^\varepsilon$  defined by the dynamics of the particle system for particles with diameter  $\varepsilon$ . To ease notation we drop the dependence on  $\varepsilon$  and write  $\hat{P}_t$ . The key result of this section is that  $\hat{P}_t$  solves the differential equation (2.4.2) which is similar to the idealised equation (2.3.1). The similarity between the two equations is exploited in the next section to prove the required convergence as  $\varepsilon$  tends to zero.

We do this by restricting our attention to collision histories that are well controlled in various ways, calling these good histories.

**Definition 2.4.1.** For a collision history  $\Phi \in \mathcal{MT}$  define  $\mathcal{V}(\Phi) \in [0, \infty)$  to be the maximum velocity involved in the history. That is,

$$\mathcal{V}(\Phi) := \max \left\{ \max_{j=1, \dots, n(\Phi)} \{|v_j|\}, \max_{s \in [0, T]} \{|v(s)|\} \right\}.$$

**Definition 2.4.2.** A collision history  $\Phi \in \mathcal{MT}$  is called *re-collision free at diameter  $\varepsilon$*  if for all  $1 \leq j \leq n(\Phi)$  and for all  $t \in [0, T] \setminus \{t_j\}$ ,

$$|x(t) - (x_j + tv_j)| > \varepsilon.$$

That is to say, if the history involves a collision between the tagged particle and background particle  $j$  at time  $t_j$  then the tagged particle has not previously collided with background particle  $j$  and the tagged particle will not re-collide with particle  $j$  up to time  $T$ . So if a history is re-collision free then it involves at most one collision per background particle. Further define

$$R(\varepsilon) := \{\Phi \in \mathcal{MT} : \Phi \text{ is re-collision free at diameter } \varepsilon\}.$$

**Definition 2.4.3.** A history  $\Phi \in \mathcal{MT}$  is called *non-grazing* if all collisions in  $\Phi$  are non-grazing, that is if,

$$\min_{1 \leq j \leq n(\Phi)} \nu_j \cdot (v(t_j^-) - v_j) > 0.$$

**Definition 2.4.4.** A history  $\Phi \in \mathcal{MT}$  is called free from initial overlap at diameter  $\varepsilon > 0$  if initially the tagged particle is at least  $\varepsilon$  away from the centre of each background particle. Explicitly if, for  $j = 1, \dots, N$ ,

$$|x_0 - x_j| > \varepsilon.$$

Define  $S(\varepsilon) \subset \mathcal{MT}$  to be the set of all histories that are free from initial overlap at radius  $\varepsilon$ .

**Definition 2.4.5.** For any pair of decreasing functions  $V, M : (0, \infty) \rightarrow [0, \infty)$  such that  $\lim_{\varepsilon \rightarrow 0} V(\varepsilon) = \lim_{\varepsilon \rightarrow 0} M(\varepsilon) = \infty$ , define the set of *good histories of diameter  $\varepsilon$* ,  $\mathcal{G}(\varepsilon)$ , by

$$\mathcal{G}(\varepsilon) := \left\{ \Phi \in \mathcal{MT} : n(\Phi) \leq M(\varepsilon), \mathcal{V}(\Phi) < V(\varepsilon), \right. \\ \left. \Phi \in R(\varepsilon) \cap S(\varepsilon) \text{ and } \Phi \text{ is non-grazing} \right\}$$

Since  $M, V$  are decreasing for  $\varepsilon' < \varepsilon$  we have  $\mathcal{G}(\varepsilon) \subset \mathcal{G}(\varepsilon')$ . Later some conditions on  $M$  and  $V$  are required to prove that  $\hat{P}_t$  solves the relevant equation and to prove convergence.

Now we define the operator  $\hat{\mathcal{Q}}_t$  which mirrors the idealised operator  $\mathcal{Q}_t$  in the empirical case. For a given history  $\Phi$ , a time  $0 < t < T$  and  $\varepsilon > 0$ , define the function  $\mathbb{1}_t^\varepsilon[\Phi] : U \times \mathbb{R}^3 \rightarrow \{0, 1\}$  by

$$\mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) := \begin{cases} 1 & \text{if for all } s \in (0, t), |x(s) - (\bar{x} + s\bar{v})| > \varepsilon, \\ 0 & \text{else.} \end{cases} \quad (2.4.1)$$

That is  $\mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v})$  is 1 if a background particle starting at the position  $(\bar{x}, \bar{v})$  avoids

colliding with the tagged particle defined by  $\Phi$  up to the time  $t$ . For a history  $\Phi$ ,  $t \geq 0$  and  $\varepsilon > 0$  define the gain operator,

$$\hat{\mathcal{Q}}_t^+[\hat{P}_t](\Phi) := \begin{cases} \delta(t - \tau) \hat{P}_t(\bar{\Phi}) \frac{g_0(v')[(v(\tau^-) - v') \cdot \nu]_+}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0. \end{cases}$$

Next define the loss operator,

$$\hat{\mathcal{Q}}_t^-[\hat{P}_t](\Phi) := \hat{P}_t(\Phi) \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}}.$$

For some  $\hat{C}(\varepsilon) > 0$  depending on  $t$  and  $\Phi$  of  $o(1)$  as  $\varepsilon$  tends to zero detailed later. Finally define the operator  $\hat{\mathcal{Q}}_t$  as follows,

$$\hat{\mathcal{Q}}_t = \hat{\mathcal{Q}}_t^+ - \hat{\mathcal{Q}}_t^-.$$

**Theorem 2.4.6.** *For  $\varepsilon$  sufficiently small and for  $\Phi \in \mathcal{G}(\varepsilon)$ ,  $\hat{P}_t$  solves the following*

$$\begin{cases} \partial_t \hat{P}_t(\Phi) &= (1 - \gamma(t)) \hat{\mathcal{Q}}_t[\hat{P}_t](\Phi) \\ \hat{P}_0(\Phi) &= \zeta(\varepsilon) f_0(x_0, v_0) \mathbb{1}_{n(\Phi)=0}. \end{cases} \quad (2.4.2)$$

The functions  $\gamma$  and  $\zeta$  are given by

$$\zeta(\varepsilon) := (1 - \frac{4}{3} \pi \varepsilon^3)^N, \quad (2.4.3)$$

and,

$$\gamma(t) := \begin{cases} n(\bar{\Phi}) \varepsilon^2 & \text{if } t = \tau, \\ n(\Phi) \varepsilon^2 & \text{if } t > \tau. \end{cases}$$

**Remark 2.4.7.** When choosing the background particles according to some Poisson point process some of these terms simplify as in [37].

The proof is developed by a series of lemmas in which we investigate the gain term, loss term and initial condition separately.

**Definition 2.4.8.** Define  $\omega_0 := (u_0, w_0) \in U \times \mathbb{R}^3$  to be the random initial position of the tagged particle. By our model  $\omega_0$  has distribution  $f_0$ .

Further for  $j = 1, \dots, N$  define  $\omega_j := (u_j, w_j)$  to be the random initial position and velocity of background particle  $j$ . Note that  $\omega_j$  has distribution  $\text{Unif}(U) \times g_0$ . Finally define  $\omega := (\omega_1, \dots, \omega_N)$ .

**Lemma 2.4.9.** *Let  $\varepsilon > 0$  and  $\Psi \in \mathcal{G}(\varepsilon)$  then  $\hat{P}_t$  is absolutely continuous with respect to the Lebesgue measure  $\lambda$  on a neighbourhood of  $\Psi$ .*

*Proof.* Recall the definition of  $B_h(\Psi)$  in (2.2.9). Since  $\mathcal{G}(\varepsilon)$  is open there exists a  $h > 0$  such that  $B_h(\Psi) \subset \mathcal{G}(\varepsilon)$ . In the case  $n(\Psi) = 0$  for all  $t \geq 0$ ,  $\hat{P}_t(\Psi) \leq f_0(x_0, v_0)$  and hence absolute continuity follows.

Suppose  $n(\Phi) \geq 1$ . Define a map  $\varphi : B_h(\Psi) \rightarrow \mathcal{MT} \times U \times \mathbb{R}^3$ ,

$$\varphi(\Phi) := (\bar{\Phi}, (x(\tau) + \varepsilon\nu - \tau v', v')).$$

We view  $\varphi$  as having  $n(\Phi) + 1$  components, the first being the initial position of the tagged particle  $(x_0, v_0)$ , components  $j = 2, \dots, n$  being the marker  $(t_j, \nu_j, \nu_j)$  and the final component being  $(x(\tau) + \varepsilon\nu - \tau v', v')$  - the initial position of the background particle that leads to the final collision with the tagged particle in  $\Phi$ . We claim that,

$$\det(\nabla\varphi)(\Phi) = \varepsilon^2(v(\tau^-) - v') \cdot \nu. \quad (2.4.4)$$

To prove this we first rotate our coordinate axis so that  $\nu = \mathbf{e}_1$ . Then for  $k = 0, \dots, n$  define  $F_{0,k} := \nabla_{x_0}\varphi_k(\Phi)$  and for  $j = 1, \dots, n$  define  $F_{j,k} := \nabla_{t_j, \nu_j}\varphi_k(\Phi)$ . We calculate,

$$F_{n+1, n+1} = \nabla_{\tau, \nu}\varphi_{n+1}(\Phi) = \begin{pmatrix} (v(\tau^-) - v') \cdot \nu & 0 & 0 \\ \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \end{pmatrix},$$

where the blank components are not needed. Also,  $F_{0,n} = \nabla_{x_0}\varphi_n(\Phi) = \text{Id}(2)$ . Further for  $j = 2, \dots, n+1$ ,  $F_{j,j} = \nabla_{t_j, \nu_j}\varphi_j(\Phi) = \text{Id}(2)$ . For any other  $j, k$  not already calculated,  $F_{j,k} = 0$ .

Hence  $\det(\nabla\varphi)(\Phi)$  is the product of the determinants of all  $F_{k,k}$  for  $k = 1, \dots, n+1$ , proving (2.4.4). Now define a second map,  $\tilde{\varphi} : B_h(\Psi) \rightarrow (U \times \mathbb{R})^{n+1}$ ,

$$\tilde{\varphi}(\Phi) := ((x_0, v_0), (x_1, v_1), \dots, (x_n, v_n)).$$

This maps  $\Phi$  to the initial position of each particle in  $\Phi$ . By repeatedly applying (2.4.4),

$$\det(\nabla\tilde{\varphi})(\Phi) = \prod_{j=1}^{n(\Phi)} (\varepsilon^2(v(t_j^-) - v_j) \cdot \nu_j). \quad (2.4.5)$$

For  $h > 0$  and  $j = 0, \dots, n$  define  $C_{h,j}(\Phi) = C_{h,j} \subset U \times \mathbb{R}^3$  to be the cube with side

length  $h$  centred at  $\tilde{\varphi}_j(\Phi)$ . Further for  $h > 0$  define,

$$C_h(\Phi) := \prod_{j=0}^n C_{h,j}(\Phi).$$

By the fact that the probability of seeing the history at time  $t$  is less than the probability that initially there is a particles at the required initial position,

$$\begin{aligned} \hat{P}_t(\tilde{\varphi}^{-1}(C_h)) &\leq \int_{C_{h,0}} f_0(x, v) \, dx \, dv \\ &\quad \times \prod_{j=1}^n N \int_{C_{h,j}} g_0(v) \, dx \, dv. \end{aligned} \quad (2.4.6)$$

Recalling that  $\lambda$  denotes the Lebesgue measure on  $\mathcal{MT}$ , by (2.4.5) it follows,

$$\lambda(\tilde{\varphi}^{-1}(C_h)) = \frac{h^{6(n+1)}}{\prod_{j=1}^n (\varepsilon^2(v(t_j^-) - v_j) \cdot \nu_j)} (1 + o(1)). \quad (2.4.7)$$

Hence combining (2.4.6) and (2.4.7) and recalling (2.2.1),

$$\begin{aligned} \frac{\hat{P}_t(\tilde{\varphi}^{-1}(C_h))}{\lambda(\tilde{\varphi}^{-1}(C_h))} &\leq \frac{1}{h^6} \int_{C_{h,0}} f_0(x, v) \, dx \, dv \\ &\quad \times \prod_{j=1}^n \left( \frac{(v(t_j^-) - v_j) \cdot \nu_j}{h^6(1 + o(1))} \int_{C_{h,j}} g_0(v) \, dx \, dv \right). \end{aligned} \quad (2.4.8)$$

Since  $f_0 \in L^1(U \times \mathbb{R}^3)$  and  $g_0 \in L^1(\mathbb{R}^3)$  let  $h$  tend to zero and the left hand side, which becomes  $\hat{P}_t(\Phi)/\lambda(\Phi)$  in the limit, remains bounded. This completes the proof.  $\square$

We now prove the initial condition requirement on  $\hat{P}_t$ .

**Lemma 2.4.10.** *Under the assumptions and set up of theorem 2.4.6 we have*

$$\hat{P}_0(\Phi) = \zeta(\varepsilon) f_0(x_0, v_0) \mathbb{1}_{n(\Phi)=0}.$$

*Proof.* In the case  $n(\Phi) > 0$ , we have  $\hat{P}_0(\Phi) = 0$ . This is because the history is free from initial overlap and involves collisions happening at some positive time therefore the collisions cannot have occurred at time 0.

Now consider  $n(\Phi) = 0$ . In this situation the history  $\Phi$  contains only the tagged particle and the probability of finding the tagged particle at the given initial data  $(x_0, v_0)$  is given by  $f_0(x_0, v_0)$ . However this must be multiplied by a factor less than one because we must rule out situations that would give initial overlap of the tagged

particle with a background particle. So we calculate the probability that there is no overlap. Firstly,

$$\begin{aligned}\mathbb{P}(|x_0 - x_1| > \varepsilon) &= 1 - \mathbb{P}(|x_0 - x_1| < \varepsilon) = 1 - \int_{\mathbb{R}^3} \int_{|x_0 - x_1| < \varepsilon} g_0(\bar{v}) \, dx_1 \, d\bar{v} \\ &= 1 - \frac{4}{3}\pi\varepsilon^3 \int_{\mathbb{R}^3} g_0(\bar{v}) \, d\bar{v} = 1 - \frac{4}{3}\pi\varepsilon^3.\end{aligned}$$

Hence,

$$\mathbb{P}(|x_0 - x_j| > \varepsilon, \forall j = 1, \dots, N) = \mathbb{P}(|x_0 - x_1| > \varepsilon)^N = (1 - \frac{4}{3}\pi\varepsilon^3)^N = \zeta(\varepsilon),$$

as required.  $\square$

To prove the loss term of (2.4.2) we first show that that rate of probability that the tagged particle collides with two background particles close to time  $t$  converges to zero. Then we calculate precisely the rate at which the tagged particle for a good tree collides with a background particle - which gives the loss rate.

**Definition 2.4.11.** Let  $\Phi \in \mathcal{G}(\varepsilon)$ . Define for  $h > 0$ ,

$$\begin{aligned}W_h(t) := \Big\{ (\bar{x}, \bar{v}) \in U \times \mathbb{R}^3 : \exists (\nu', t') \in \mathbb{S}^2 \times (t, t+h) \\ \text{such that } x(t') + \varepsilon\nu' = \bar{x} + t'\bar{v} \text{ and } (v(t'^-) - \bar{v}) \cdot \nu' > 0 \Big\}.\end{aligned}$$

That is  $W_h(t)$  is the set of initial points in  $U \times \mathbb{R}^3$  for the background particles that lead to a collision with the tagged particle as defined by  $\Phi$  between the time  $t$  and  $t+h$ .

Now consider  $\hat{P}_t((x_1, v_1) \in W_h(t) \mid \Phi)$ , the probability that the background particle with label 1 has initial position and velocity in  $W_h(t)$  given that at time  $t$   $\Phi$  exists. If we consider a good history  $\Phi$ , a given  $\varepsilon > 0$  and assume that background particle 1 is not involved in the history then, since we are conditioning on  $\Phi$  existing at the time  $t$ , we know that the background particle cannot have started in the region where  $\mathbb{1}_t^\varepsilon[\Phi]$  is zero. Hence conditioning on  $\Phi$  existing at time  $t$  the initial distribution of background particle 1 that is not involved in the history is,

$$\frac{g_0(v_1) \mathbb{1}_t^\varepsilon[\Phi](x_1, v_1)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}}.$$

Hence it follows that

$$\hat{P}_t((x_1, v_1) \in W_h(t) \mid \Phi) = \frac{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_{W_h(t)}(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} =: I_h(t). \quad (2.4.9)$$

To prove the loss term of (2.4.2) we calculate explicitly the rate of change of  $\hat{P}_t^\varepsilon(\Phi)$  for a given good history  $\Phi$ . We show that the effect from seeing at least two collisions converges to zero and that to calculate the effect from exactly one collision we calculate the limit of  $I_h(t)$  as  $h$  tends to zero.

From now on assume that the functions  $V$  and  $M$  in definition 2.4.5 satisfy, for any  $0 < \varepsilon < 1$ ,

$$\varepsilon V(\varepsilon)^3 \leq \frac{1}{8}, \quad (2.4.10)$$

and,

$$M(\varepsilon) \leq \frac{1}{\sqrt{\varepsilon}}. \quad (2.4.11)$$

**Lemma 2.4.12.** *Recall definition 2.4.8. For  $\varepsilon$  sufficiently small,  $\Phi \in \mathcal{G}(\varepsilon)$  and  $t > \tau$ ,*

$$\lim_{h \downarrow 0} \frac{1}{h} \hat{P}_t(\#(\omega \cap W_h(t)) \geq 2 \mid \Phi) = 0, \quad (2.4.12)$$

and,

$$\lim_{h \downarrow 0} \frac{1}{h} \hat{P}_{t-h}(\#(\omega \cap W_h(t-h)) \geq 2 \mid \Phi) = 0. \quad (2.4.13)$$

*Proof.* We first prove (2.4.12). Since background particles are independent and since we are conditioning on a good history  $\Phi$  we know that  $n(\Phi)$  background particles cannot re-collide with the tagged particle,

$$\begin{aligned} & \hat{P}_t(\#(\omega \cap W_h(t)) \geq 2 \mid \Phi) \\ & \leq \hat{P}_t \left( \bigcup_{1 \leq i < j \leq N-n(\Phi)} \{(x_i, v_i) \in W_h(t) \text{ and } (x_j, v_j) \in W_h(t)\} \mid \Phi \right) \\ & \leq \sum_{1 \leq i < j \leq N-n(\Phi)} \hat{P}_t((x_i, v_i) \in W_h(t) \text{ and } (x_j, v_j) \in W_h(t) \mid \Phi) \\ & \leq N(N-1) \hat{P}_t((x_1, v_1) \in W_h(t) \text{ and } (x_2, v_2) \in W_h(t) \mid \Phi) \\ & = N(N-1) \hat{P}_t((x_1, v_1) \in W_h(t) \mid \Phi)^2. \end{aligned} \quad (2.4.14)$$

Recalling (2.4.9),

$$\hat{P}_t((x_1, v_1) \in W_h(t) \mid \Phi) = I_h(t). \quad (2.4.15)$$

Now we estimate the right hand side of (2.4.15) by estimating the numerator and denominator. Firstly by calculating the volume of the appropriate cylinder, for any  $\bar{v} \in \mathbb{R}^3$ ,

$$\int_U \mathbb{1}_{W_h(t)}(\bar{x}, \bar{v}) d\bar{x} \leq \pi \varepsilon^2 \int_t^{t+h} |v(s) - \bar{v}| ds. \quad (2.4.16)$$

Define

$$\beta := \int_{\mathbb{R}^3} g_0(v)(1 + |v|) dv. \quad (2.4.17)$$

Note that by assumption (2.2.3),  $\beta < \infty$ . Since  $\Phi \in \mathcal{G}(\varepsilon)$  it follows that  $|v(t)| \leq \mathcal{V}(\Phi) \leq V(\varepsilon)$ . Using these and (2.4.16) we estimate the numerator in (2.4.15),

$$\begin{aligned} \int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_{W_h(t)}(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v} &\leq \int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_{W_h(t)}(\bar{x}, \bar{v}) d\bar{x} d\bar{v} \\ &\leq \int_{\mathbb{R}^3} g_0(\bar{v}) \pi \varepsilon^2 \int_t^{t+h} |v(s) - \bar{v}| ds d\bar{v} \\ &\leq \pi \varepsilon^2 \int_{\mathbb{R}^3} g_0(\bar{v}) \int_t^{t+h} |v(s)| + |\bar{v}| ds d\bar{v} \\ &\leq \pi \varepsilon^2 \int_{\mathbb{R}^3} g_0(\bar{v}) \int_t^{t+h} V(\varepsilon) + |\bar{v}| ds d\bar{v} \\ &\leq \pi \varepsilon^2 \int_{\mathbb{R}^3} g_0(\bar{v}) h (V(\varepsilon) + |\bar{v}|) d\bar{v} \\ &\leq h \pi \varepsilon^2 \int_{\mathbb{R}^3} g_0(\bar{v}) (V(\varepsilon) + |\bar{v}|) d\bar{v} \leq h \pi \varepsilon^2 (V(\varepsilon) + \beta). \end{aligned} \quad (2.4.18)$$

Turning to the denominator of (2.4.9). Firstly note that,

$$\begin{aligned} \int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v} &= \int_{U \times \mathbb{R}^3} g_0(\bar{v}) (1 - \mathbb{1}_{W_t(0)}(\bar{x}, \bar{v})) d\bar{x} d\bar{v} \\ &= 1 - \int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_{W_t(0)}(\bar{x}, \bar{v}) d\bar{x} d\bar{v}. \end{aligned} \quad (2.4.19)$$

By using (2.4.16),  $t \leq T$  and the same estimates from the numerator estimate,

$$\begin{aligned} \int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_{W_t(0)}(\bar{x}, \bar{v}) d\bar{x} d\bar{v} &\leq \pi \varepsilon^2 \int_{\mathbb{R}^3} g_0(\bar{v}) \int_0^t |v(s) - \bar{v}| ds d\bar{v} \\ &\leq \pi \varepsilon^2 \int_{\mathbb{R}^3} g_0(\bar{v}) \int_0^t \mathcal{V}(\Phi) + |\bar{v}| ds d\bar{v} \\ &\leq \pi \varepsilon^2 \int_{\mathbb{R}^3} g_0(\bar{v}) t (V(\varepsilon) + |\bar{v}|) d\bar{v} \leq \pi \varepsilon^2 T (V(\varepsilon) + \beta). \end{aligned} \quad (2.4.20)$$

Hence for  $\varepsilon$  sufficiently small by (2.4.10),

$$\pi \varepsilon^2 T (V(\varepsilon) + \beta) \leq 1/2,$$



so by (2.4.19) and (2.4.20),

$$\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v} \geq 1/2. \quad (2.4.21)$$

Bounds for both the numerator and the denominator of (2.4.15) have been found in equations (2.4.18) and (2.4.21) respectively. Hence,

$$I_h(t) \leq 2h\pi\varepsilon^2(V(\varepsilon) + \beta). \quad (2.4.22)$$

Substituting this into (2.4.14) and recalling (2.2.1),

$$\begin{aligned} \hat{P}_t(\#(\omega \cap W_h(t)) \geq 2 \mid \Phi) &\leq N(N-1) \times 4h^2\pi^2\varepsilon^4(V(\varepsilon) + \beta)^2 \\ &\leq 4h^2\pi^2N^2\varepsilon^4(V(\varepsilon) + \beta)^2 \leq 4h^2\pi^2(V(\varepsilon) + \beta)^2. \end{aligned}$$

The required result follows. For (2.4.13) we take  $h > 0$  sufficiently small so that  $t - h > \tau$ . Analogously it follows that,

$$\hat{P}_{t-h}(\#(\omega \cap W_h(t-h)) \geq 2 \mid \Phi) \leq 4h^2\pi^2(V(\varepsilon) + \beta)^2,$$

completing the proof of the lemma.  $\square$

The previous lemma will be used to show that the rate of change of  $\hat{P}_t(\Phi)$  caused by the tagged particle seeing at least two collisions is zero. We now calculate the effect of exactly one collision. Before we do this we first estimate the error caused by re-collisions.

**Definition 2.4.13.** For  $\Phi \in \mathcal{G}(\varepsilon)$ ,  $t > \tau$  and  $h > 0$  recall the definition of  $W_h(t)$  in definition 2.4.11 and  $\mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v})$  (2.4.1). Define,  $B_{h,t}(\Phi) \subset U \times \mathbb{R}^3$ ,

$$B_{h,t}(\Phi) := \{(\bar{x}, \bar{v}) \in U \times \mathbb{R}^3 : \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) = 0 \text{ and } \mathbb{1}_{W_h(t)}(\bar{x}, \bar{v}) = 1\}.$$

Notice that  $B_{h,t}(\Phi)$  is the set of all initial positions that a background particle can take such that it collides with the tagged particle once during  $(0, t)$  and once during  $(t, t+h)$ .

**Lemma 2.4.14.** For  $\varepsilon$  sufficiently small,  $\Phi \in \mathcal{G}(\varepsilon)$ ,  $t > \tau$  and  $h > 0$  sufficiently small there exists a  $\hat{C}(\varepsilon) > 0$  depending on  $t$  and  $\Phi$  with  $\hat{C}(\varepsilon) = o(1)$  as  $\varepsilon$  tends to zero such that,

$$\int_{B_{h,t}(\Phi)} g_0(\bar{v}) \, d\bar{x} \, d\bar{v} = h\varepsilon^2 \hat{C}(\varepsilon) = \int_{B_{h,t-h}(\Phi)} g_0(\bar{v}) \, d\bar{x} \, d\bar{v}.$$

*Proof.* We first prove the first equality. Recall that  $(x(t), v(t))$  is the trajectory of the tagged particle defined by the history  $\Phi$ .  $B_{h,t}(\Phi)$  is given by,

$$\begin{aligned} B_{h,t}(\Phi) = \{(\bar{x}, \bar{v}) : \exists s \in (0, t), \sigma \in (t, t+h), \nu_1, \nu_2 \in \mathbb{S}^2 \text{ such that} \\ \bar{x} + s\bar{v} - x(s) = \varepsilon\nu_1, \bar{x} + \sigma\bar{v} - x(\sigma) = \varepsilon\nu_2 \text{ and} \\ (v(s) - \bar{v}) \cdot \nu_1 > 0, (v(\sigma) - \bar{v}) \cdot \nu_2 > 0\}. \end{aligned}$$

Define  $\delta := \varepsilon^{1/3}$ . We split the set  $B_{h,t}(\Phi)$  into two parts, the first, denoted  $B_{h,t}^\delta(\Phi)$ , which considers  $s \in (0, t - \delta]$  and the second, denoted  $B_{h,t}^2(\Phi)$ , which considers  $s \in (t - \delta, t)$ . We evaluate the bounds on these two sets separately.

Consider  $s \in (0, t - \delta]$  and  $\sigma \in (t, t+h)$  fixed. The conditions defined by  $B_{h,t}$  require that,

$$\bar{v} = \frac{x(\sigma) - x(s)}{\sigma - s} + \frac{\varepsilon\nu_2 - \varepsilon\nu_1}{\sigma - s}.$$

For  $\sigma$  fixed this implies that  $\bar{v}$  is contained in a cylinder of radius  $2\varepsilon/\delta$  around the curve defined by  $\frac{x(\sigma) - x(s)}{\sigma - s}$  for  $s \in (0, t - \delta]$ . Recalling the definition of  $V(\varepsilon)$  from definition 2.4.5, taking  $h \ll \varepsilon/V(\varepsilon)$  implies  $h|v(\sigma)| \ll \varepsilon$  so the dependence on  $\sigma \in (t, t+h)$  gives only a small perturbation around the curve defined by  $\frac{x(t) - x(s)}{t - s}$  for  $s \in (0, t - \delta]$ . Hence  $\bar{v}$  is contained in the cylinder with radius  $4\varepsilon/\delta$  around the piecewise differentiable curve  $r(s) := \frac{x(t) - x(s)}{t - s}$  for  $s \in (0, t - \delta]$ .

Denote this cylinder in  $\mathbb{R}^3$  by  $E = E(t, \Phi, \delta)$ . We seek a bound on the volume of  $E$ ,  $|E|$ . First consider the length of the curve  $r$ . For almost all  $s \in (0, t - \delta)$ ,

$$\frac{d}{ds}r(s) = \frac{x(t) - x(s)}{(t - s)^2} + \frac{v(s)}{t - s}.$$

Hence,

$$\left| \frac{d}{ds}r(s) \right| \leq \frac{|x(t) - x(s)|}{(t - s)^2} + \frac{|v(s)|}{t - s} \leq \frac{3}{(t - s)^2} + \frac{V(\varepsilon)}{t - s}.$$

Thus the length of the curve is bounded by,

$$\int_0^{t-\delta} \left| \frac{d}{ds}r(s) \right| ds \leq \int_0^{t-\delta} \frac{3}{(t - s)^2} + \frac{V(\varepsilon)}{t - s} ds = \frac{3}{\delta} - \frac{3}{t} - V(\varepsilon)(\log(\delta) - \log(t)).$$

Therefore for some  $C > 0$ ,

$$|E| \leq C \left( \frac{\varepsilon}{\delta} \right)^2 \left( \frac{3}{\delta} - \frac{3}{t} - V(\varepsilon)(\log(\delta) - \log(t)) \right). \quad (2.4.23)$$

Noting that  $x(\sigma) = x(t) + (\sigma - t)v(t)$ , for  $\bar{v}$  given,  $(\bar{x}, \bar{v}) \in B_{h,t}(\Phi)$  requires that,

$$\begin{aligned}\bar{x} &= x(\sigma) - \sigma\bar{v} + \varepsilon\nu_2 = x(t) + (\sigma - t)v(t) - \sigma\bar{v} + \varepsilon\nu_2 \\ &= x(t) - t\bar{v} + (\sigma - t)(v(t) - \bar{v}) + \varepsilon\nu_2.\end{aligned}$$

Hence for  $\bar{v}$  given  $\bar{x}$  is contained in cylinder of radius  $\varepsilon$  and length  $h|v(t) - \bar{v}|$ . Denote this cylinder by  $C(\bar{v})$ . By (2.2.4) or (2.2.5), for constants  $C$  that may change on each line,

$$\begin{aligned}\int_{B_{h,t}^\delta(\Phi)} g_0(\bar{v}) d\bar{x} d\bar{v} &\leq \int_E g_0(\bar{v}) \int_{C(\bar{v})} d\bar{x} d\bar{v} \leq C\varepsilon^2 h \int_E g_0(\bar{v}) |v(t) - \bar{v}| d\bar{v} \\ &\leq C\varepsilon^2 h \int_E g_0(\bar{v}) (V(\varepsilon) + |\bar{v}|) d\bar{v} \leq C\varepsilon^2 h (V(\varepsilon) + 1) |E|.\end{aligned}$$

It remains to show that  $(V(\varepsilon) + 1)|E|$  is  $o(1)$  as  $\varepsilon$  tends to zero. Recall (2.4.10), that  $\delta = \varepsilon^{1/3}$  and (2.4.23),

$$\begin{aligned}(V(\varepsilon) + 1)|E| &= (V(\varepsilon) + 1) \times C \left( \frac{\varepsilon}{\delta} \right)^2 \left( \frac{3}{\delta} - \frac{3}{t} - V(\varepsilon)(\log(\delta) - \log(t)) \right) \\ &\leq C \left( \frac{1}{2\varepsilon^{1/3}} + 1 \right) \varepsilon^{4/3} \left( \frac{3}{\varepsilon^{1/3}} + \frac{1}{2\varepsilon^{1/3}} (|\log \varepsilon^{1/3}| + |\log t|) \right) \\ &\leq C \left( \frac{1}{2\varepsilon^{1/3}} + 1 \right) \left( 3\varepsilon + \frac{1}{2}\varepsilon \left( \frac{1}{3} |\log \varepsilon| + |\log t| \right) \right)\end{aligned}$$

as required.

Now consider the second part of  $B_{h,t}(\Phi)$  for  $s \in (t - \delta, t)$  denoted  $B_{h,t}^2(\Phi)$ . Since  $\Phi$  is fixed,  $t > \tau$  and  $\delta = \varepsilon^{1/3}$  let  $\varepsilon$  sufficiently small such that  $t - \delta > \tau$ . Hence for  $s \in (t - \delta, t)$ ,  $v(s) = v(t)$ . We change the velocity space coordinates so that  $v(t) = 0$ . If we require that a particle starting at  $(\bar{x}, \bar{v})$  collides with the tagged particle in  $(t - \delta, t)$  and again in  $(t, t + h)$  we require in the new coordinates that either  $\bar{v} = 0$  or that  $|\bar{v}|$  is sufficiently large so that the background particle wraps round the torus having travelled at least distance  $3/4$  (for  $\varepsilon$  sufficiently small) within time  $(\delta + h)$  to re-collide with the tagged particle. That is,

$$|\bar{v}| \geq \frac{3}{4(\delta + h)}.$$

For  $h \leq \delta/4$ , this implies  $|\bar{v}| \geq \frac{3}{5\delta}$ . Changing back to the original coordinates, this means it is required that  $\bar{v} = v(t)$  or  $|\bar{v} - v(t)| \geq \frac{3}{5\delta}$ .

For a given  $\bar{v}$ , the same conditions as before on the  $\bar{x}$  coordinate must hold and so  $\bar{x}$  is in the cylinder  $C(\bar{v})$ . Recalling (2.2.3), (2.4.10),  $\delta = \varepsilon^{1/3}$  and that  $|v(t)| \leq V(\varepsilon) \leq$

$1/2\varepsilon^{-1/3} = \frac{1}{2\delta}$  it follows for constants  $C$  that change on each line,

$$\begin{aligned}
\int_{B_{h,t}^2(\Phi)} g_0(\bar{v}) \, d\bar{x} \, d\bar{v} &\leq \int_{\mathbb{R}^3 \setminus B_{3/5\delta}(v(t))} g_0(\bar{v}) \int_{C(\bar{v})} d\bar{x} \, d\bar{v} \\
&\leq C\varepsilon^2 h \int_{\mathbb{R}^3 \setminus B_{3/5\delta}(v(t))} g_0(\bar{v}) |v(t) - \bar{v}| \, d\bar{v} \\
&\leq C\varepsilon^2 h \int_{\mathbb{R}^3 \setminus B_{3/5\delta}(v(t))} g_0(\bar{v}) (V(\varepsilon) + |\bar{v}|) \, d\bar{v} \\
&\leq C\varepsilon^2 h \int_{\mathbb{R}^3 \setminus B_{1/10\delta}(0)} g_0(\bar{v}) (V(\varepsilon) + |\bar{v}|) \, d\bar{v} \\
&\leq C\varepsilon^2 h \int_{\mathbb{R}^3 \setminus B_{1/10\delta}(0)} g_0(\bar{v}) (100\delta^2 |\bar{v}|^2 V(\varepsilon) + 10\delta |\bar{v}|^2) \, d\bar{v} \\
&\leq C\varepsilon^2 h (10\delta^2 V(\varepsilon) + \delta) \\
&\leq C\varepsilon^2 h \left( 5 \frac{\varepsilon^{2/3}}{\varepsilon^{1/3}} + \varepsilon^{1/3} \right) = C\varepsilon^2 h \times \varepsilon^{1/3},
\end{aligned}$$

as required. Since  $B_{h,t}(\Phi) = B_{h,t}^\delta(\Phi) \cup B_{h,t}^2(\Phi)$  the proof of the first equality is complete. For the second equality we simply take  $h$  sufficiently small so that  $t - h > \tau$  and apply a similar argument. □

**Lemma 2.4.15.** *For  $\varepsilon$  sufficiently small,  $\Phi \in \mathcal{G}(\varepsilon)$ ,  $t > \tau$  and  $\hat{C}(\varepsilon)$  as in the above lemma,*

$$\begin{aligned}
\lim_{h \downarrow 0} \frac{1}{h} \hat{P}_t(\#(\omega \cap W_h(t)) = 1 \mid \Phi) \\
= (1 - \gamma(t)) \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(t) - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{I}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}}, \tag{2.4.24}
\end{aligned}$$

and,

$$\begin{aligned}
\lim_{h \downarrow 0} \frac{1}{h} \hat{P}_{t-h}(\#(\omega \cap W_h(t-h)) = 1 \mid \Phi) \\
= (1 - \gamma(t)) \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(t) - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{I}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}}, \tag{2.4.25}
\end{aligned}$$

*Proof.* We first prove (2.4.24). Since the initial data for each background particle is

independent of the other background particles,

$$\begin{aligned}
\hat{P}_t(\#(\omega \cap W_h(t)) = 1 \mid \Phi) &= \sum_{i=1}^{N-n(\Phi)} \hat{P}_t\left((x_i, v_i) \in W_h(t) \text{ and } (x_1, v_1), \dots, \right. \\
&\quad \left. (x_{i-1}, v_{i-1}), (x_{i+1}, v_{i+1}), \dots, (x_{N-n(\Phi)}, v_{N-n(\Phi)}) \notin W_h(t) \mid \Phi\right) \\
&= (N - n(\Phi)) \hat{P}_t((x_1, v_1) \in W_h(t) \mid \Phi) \hat{P}_t((x_2, v_2) \notin W_h(t) \mid \Phi)^{N-n(\Phi)-1} \\
&= (N - n(\Phi)) I_h(t) (1 - I_h(t))^{N-n(\Phi)-1} \\
&= (N - n(\Phi)) I_h(t) \sum_{j=0}^{N-n(\Phi)-1} (-1)^j \binom{N - n(\Phi) - 1}{j} I_h(t)^j \\
&= (N - n(\Phi)) \sum_{j=0}^{N-n(\Phi)-1} (-1)^j \binom{N - n(\Phi) - 1}{j} I_h(t)^{j+1}. \tag{2.4.26}
\end{aligned}$$

By (2.4.22),

$$\lim_{h \rightarrow 0} \frac{1}{h} I_h(t)^2 = 0.$$

Hence dividing (2.4.26) by  $h$  and taking  $h$  to zero we see that all terms in the sum for  $j \geq 1$  tend to zero, leaving only the contribution from the term  $j = 0$ . Hence,

$$\begin{aligned}
\lim_{h \rightarrow 0} \frac{1}{h} \hat{P}_t(\#(\omega \cap W_h(t)) = 1 \mid \Phi) &= \lim_{h \rightarrow 0} \frac{1}{h} (N - n(\Phi)) \sum_{j=0}^{N-n(\Phi)-1} (-1)^j \binom{N - n(\Phi) - 1}{j} I_h(t)^{j+1} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} (N - n(\Phi)) I_h(t). \tag{2.4.27}
\end{aligned}$$

It remains to investigate,

$$\lim_{h \rightarrow 0} \frac{1}{h} I_h(t) = \lim_{h \rightarrow 0} \frac{1}{h} \frac{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_{W_h(t)}(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}}.$$

For  $B_{h,t}(\Phi)$  as defined in definition 2.4.13 we have,

$$\begin{aligned}
&\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_{W_h(t)}(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v} \\
&= \int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_{W_h(t)}(\bar{x}, \bar{v}) \, d\bar{v} - \int_{B_{h,t}(\Phi)} g_0(\bar{v}) \, d\bar{x} \, d\bar{v}
\end{aligned}$$

It then follows from lemma 2.4.14,

$$\begin{aligned}
\lim_{h \rightarrow 0} \frac{1}{h} I_h(t) &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_{W_h(t)}(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{h\varepsilon^2 \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(t) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu - h\varepsilon^2 \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} \right) \\
&= \varepsilon^2 \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(t) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}}.
\end{aligned}$$

Substituting this into (2.4.27),

$$\begin{aligned}
\lim_{h \rightarrow 0} \frac{1}{h} \hat{P}_t(\#(\omega \cap W_h(t)) = 1 \mid \Phi) &= \lim_{h \rightarrow 0} \frac{1}{h} (N - n(\Phi)) I_h(t) \\
&= (N - n(\Phi)) \varepsilon^2 \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(t) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} \\
&= (1 - \gamma(t)) \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(t) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}},
\end{aligned}$$

which proves (2.4.24). For (2.4.25) we similarly prove that,

$$\lim_{h \rightarrow 0} \frac{1}{h} \hat{P}_{t-h}(\#(\omega \cap W_h(t-h)) = 1 \mid \Phi) = \lim_{h \rightarrow 0} \frac{1}{h} (N - n(\Phi)) I_h(t-h).$$

and again use lemma 2.4.14.  $\square$

Before we can prove the loss term of (2.4.2) we it remains to show that  $\hat{P}_t(\Phi)$  is continuous with respect to  $t$ .

**Lemma 2.4.16.** *For  $\varepsilon$  sufficiently small,  $\Phi \in \mathcal{G}(\varepsilon)$ ,  $\hat{P}(\Phi) : (\tau, T] \rightarrow [0, \infty)$  is continuous.*

*Proof.* Let  $t \in (\tau, T]$ . Then for  $h > 0$  we have,

$$\hat{P}_{t+h}(\Phi) = (1 - \hat{P}_t(\#(\omega \cap W_h(t)) > 0 \mid \Phi)) \hat{P}_t(\Phi). \quad (2.4.28)$$

It follows by (2.4.12) and (2.4.24) that,

$$\begin{aligned}
|\hat{P}_{t+h}(\Phi) - \hat{P}_t(\Phi)| &= \hat{P}_t(\#(\omega \cap W_h(t)) > 0 \mid \Phi) \hat{P}_t(\Phi) \\
&\rightarrow 0 \text{ as } h \rightarrow 0.
\end{aligned}$$

Now let  $h > 0$  sufficiently small so that  $t-h > \tau$ . Then,

$$\hat{P}_t(\Phi) = (1 - \hat{P}_{t-h}(\#(\omega \cap W_h(t-h)) > 0 \mid \Phi)) \hat{P}_{t-h}(\Phi). \quad (2.4.29)$$

We now note that  $\hat{P}_{t-h}(\Phi) \leq \hat{P}_\tau(\Phi)$ , since there is a non-negative probability that the tagged particle experiences a collision in the time  $[\tau, t-h]$ . Hence by (2.4.29), (2.4.13) and (2.4.25),

$$\begin{aligned} |\hat{P}_t(\Phi) - \hat{P}_{t-h}(\Phi)| &= \hat{P}_{t-h}(\#(\omega \cap W_h(t-h)) > 0 \mid \Phi) \hat{P}_{t-h}(\Phi) \\ &\leq \hat{P}_{t-h}(\#(\omega \cap W_h(t-h)) > 0 \mid \Phi) \hat{P}_\tau(\Phi) \\ &\rightarrow 0 \text{ as } h \rightarrow 0. \end{aligned}$$

□

With this we can now prove the loss term of theorem 2.4.6.

**Lemma 2.4.17.** *Under the assumptions and set up of theorem 2.4.6, for  $t > \tau$ ,*

$$\partial_t \hat{P}_t(\Phi) = (1 - \gamma(t)) \hat{\mathcal{Q}}_t^-[\hat{P}_t](\Phi).$$

*Proof.* By (2.4.28), (2.4.12) and (2.4.24),

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{h} \left( \hat{P}_{t+h}(\Phi) - \hat{P}_t(\Phi) \right) &= - \lim_{h \rightarrow 0} \frac{1}{h} \hat{P}_t(\#(\omega \cap W_h(t)) > 0 \mid \Phi) \hat{P}_t(\Phi) \\ &= - \hat{P}_t(\Phi) \lim_{h \rightarrow 0} \frac{1}{h} \hat{P}_t(\#(\omega \cap W_h(t)) = 1 \mid \Phi) \\ &= - \hat{P}_t(\Phi) (1 - \gamma(t)) \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(t) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} \\ &= (1 - \gamma(t)) \hat{\mathcal{Q}}_t^-[\hat{P}_t](\Phi). \end{aligned}$$

And by (2.4.29), (2.4.13), (2.4.25) and lemma 2.4.16

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{h} \left( \hat{P}_t(\Phi) - \hat{P}_{t-h}(\Phi) \right) &= - \lim_{h \rightarrow 0} \frac{1}{h} \hat{P}_{t-h}(\#(\omega \cap W_h(t-h)) > 0 \mid \Phi) \hat{P}_{t-h}(\Phi) \\ &= - \hat{P}_t(\Phi) \lim_{h \rightarrow 0} \frac{1}{h} \hat{P}_{t-h}(\#(\omega \cap W_h(t-h)) = 1 \mid \Phi) \\ &= - \hat{P}_t(\Phi) (1 - \gamma(t)) \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(t) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} \\ &= (1 - \gamma(t)) \hat{\mathcal{Q}}_t^-[\hat{P}_t](\Phi). \end{aligned}$$

which proves the lemma. □

We next move to proving the gain term in (2.4.2).

**Remark 2.4.18.** For a probability measure  $P$  on a domain  $\Omega$  and a list of  $m \in \mathbb{N}$  measurable sets  $A_1, \dots, A_m \subset \Omega$  the inclusion exclusion principle [7, (2.9)] states that,

$$P(\cup_{i=1}^m A_i) = \sum_{i=1}^m P(A_i) - \sum_{1 \leq i < j \leq m} P(A_i \cap A_j) + \dots + (-1)^{m+1} P(\cap_{i=1}^m A_i).$$

This implies in particular,

$$P(\cup_{i=1}^m A_i) \geq \sum_{i=1}^m P(A_i) - \sum_{1 \leq i < j \leq m} P(A_i \cap A_j).$$

**Lemma 2.4.19.** *Under the assumptions and set up of theorem 2.4.6, for  $n(\Phi) \geq 1$*

$$\hat{P}_\tau(\Phi) = (1 - \gamma(\tau)) \hat{P}_\tau(\bar{\Phi}) \frac{g_0(v')[(v(\tau^-) - v') \cdot \nu]_+}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}}.$$

*Proof.* Firstly,

$$\hat{P}_\tau(\Phi) = \hat{P}_\tau(\Phi \cap \bar{\Phi}) = \hat{P}_\tau(\Phi | \bar{\Phi}) \hat{P}_\tau(\bar{\Phi}).$$

It remains to show that

$$\hat{P}_\tau(\Phi | \bar{\Phi}) = (1 - \gamma(\tau)) \frac{g_0(v')[(v(\tau^-) - v') \cdot \nu]_+}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}}. \quad (2.4.30)$$

We do this by proving upper and lower bounds. For  $h \geq 0$  define,

$$U_h := \{\Psi \in \mathcal{MT} : \bar{\Psi} = \bar{\Phi} \text{ and } \Psi \in B_h(\Phi)\}.$$

Note that  $U_0 := \{\Phi\}$ . Then by lemma 2.4.9,

$$\hat{P}_\tau(\Phi | \bar{\Phi}) = \lim_{h \rightarrow 0} h^{-6} \hat{P}_\tau(U_h | \bar{\Phi}). \quad (2.4.31)$$

For  $\Psi \in U_h$  define  $V_h(\Psi) \in U \times \mathbb{R}^3$  to be the initial position of the background particle that leads to the final collision of  $\Psi$  and define  $V_h \subset U \times \mathbb{R}^3$  by,

$$V_h = \cup_{\Psi \in U_h} V_h(\Psi).$$

Note that  $V_0 = \{(x(\tau) + \varepsilon\nu - \tau v', v')\}$  that is  $V_0$  contains only the initial point of the background particle that gives the final collision in  $\Phi$ . Then by a change of coordinates,



recalling (2.4.4),

$$\begin{aligned}
\hat{P}_\tau(U_h | \bar{\Phi}) &\leq \sum_{i=1}^{N-(n(\Phi)-1)} \hat{P}_\tau((x_i, v_i) \in V_h | \bar{\Phi}_\tau) \varepsilon^2 [(v(\tau^-) - v') \cdot \nu]_+ \\
&= (N - (n(\Phi) - 1)) \varepsilon^2 \hat{P}_\tau((x_1, v_1) \in V_h | \bar{\Phi}) [(v(\tau^-) - v') \cdot \nu]_+ \\
&= (1 - \gamma(\tau)) \hat{P}_\tau((x_1, v_1) \in V_h | \bar{\Phi}) [(v(\tau^-) - v') \cdot \nu]_+. \tag{2.4.32}
\end{aligned}$$

By absolute continuity of  $\hat{P}_\tau$ ,

$$\lim_{h \rightarrow 0} h^{-6} \hat{P}_\tau((x_1, v_1) \in V_h | \bar{\Phi}) = \hat{P}_\tau((x_1, v_1) \in V_0 | \bar{\Phi}).$$

Combining these into (2.4.31),

$$\begin{aligned}
\hat{P}_\tau(\Phi | \bar{\Phi}) &= \lim_{h \rightarrow 0} h^{-6} \hat{P}_\tau(U_h | \bar{\Phi}) \\
&\leq \lim_{h \rightarrow 0} h^{-6} (1 - \gamma(\tau)) \hat{P}_\tau((x_1, v_1) \in V_h | \bar{\Phi}) [(v(\tau^-) - v') \cdot \nu]_+ \\
&= (1 - \gamma(\tau)) \hat{P}_\tau((x_1, v_1) \in V_0 | \bar{\Phi}) [(v(\tau^-) - v') \cdot \nu]_+. \tag{2.4.33}
\end{aligned}$$

Next consider the lower bound. By the inclusion-exclusion principle as described in remark 2.4.18,

$$\begin{aligned}
\hat{P}_\tau(U_h | \bar{\Phi}) &\geq \sum_{i=1}^{N-(n(\Phi)-1)} \hat{P}_\tau((x_i, v_i) \in V_h | \bar{\Phi}) \varepsilon^2 [(v(\tau^-) - v') \cdot \nu]_+ \\
&\quad - \sum_{1 \leq i < j \leq N-(n(\Phi)-1)} \hat{P}_\tau((x_i, v_i), (x_j, v_j) \in V_h | \bar{\Phi}) \varepsilon^2 [(v(\tau^-) - v') \cdot \nu]_+. \tag{2.4.34}
\end{aligned}$$

As in (2.4.32) it follows,

$$\begin{aligned}
&\sum_{i=1}^{N-(n(\Phi)-1)} \hat{P}_\tau((x_i, v_i) \in V_h | \bar{\Phi}) \varepsilon^2 [(v(\tau^-) - v') \cdot \nu]_+ \\
&= (1 - \gamma(\tau)) \hat{P}_\tau((x_1, v_1) \in V_h | \bar{\Phi}) [(v(\tau^-) - v') \cdot \nu]_+.
\end{aligned}$$

Further,

$$\begin{aligned}
& \sum_{1 \leq i < j \leq N-(n(\Phi)-1)} \hat{P}_\tau((x_i, v_i), (x_j, v_j) \in V_h | \bar{\Phi}) \varepsilon^2 [(v(\tau^-) - v') \cdot \nu]_+ \\
& \leq N(N-1) \hat{P}_\tau((x_1, v_1), (x_2, v_2) \in V_h | \bar{\Phi}) \varepsilon^2 [(v(\tau^-) - v') \cdot \nu]_+ \\
& = (N-1) \hat{P}_\tau((x_1, v_1) \in V_h | \bar{\Phi})^2 [(v(\tau^-) - v') \cdot \nu]_+.
\end{aligned}$$

By the absolute continuity of  $\hat{P}_t$  this implies,

$$\lim_{h \rightarrow 0} \frac{1}{h^6} \sum_{1 \leq i < j \leq N-(n(\Phi)-1)} \hat{P}_\tau((x_i, v_i), (x_j, v_j) \in V_h | \bar{\Phi}) \varepsilon^2 [(v(\tau^-) - v') \cdot \nu]_+ = 0.$$

Hence by (2.4.34),

$$\begin{aligned}
\hat{P}_\tau(\Phi | \bar{\Phi}) &= \lim_{h \rightarrow 0} h^{-6} \hat{P}_\tau(U_h | \bar{\Phi}) \\
&\geq \lim_{h \rightarrow 0} h^{-6} (1 - \gamma(\tau)) \hat{P}_\tau((x_1, v_1) \in V_h | \bar{\Phi}) [(v(\tau^-) - v') \cdot \nu]_+ \\
&= (1 - \gamma(\tau)) \hat{P}_\tau((x_1, v_1) \in V_0 | \bar{\Phi}) [(v(\tau^-) - v') \cdot \nu]_+. \tag{2.4.35}
\end{aligned}$$

Recalling that we need to prove (2.4.30) to prove the lemma, we see that with (2.4.33) and (2.4.35) we now need only to show that,

$$\hat{P}_\tau((x_1, v_1) \in V_0 | \bar{\Phi}) = \frac{g_0(v')}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}}.$$

That is,

$$\hat{P}_\tau((x_1, v_1) = (x(\tau) + \varepsilon\nu - \tau v', v') | \bar{\Phi}) = \frac{g_0(v')}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}}. \tag{2.4.36}$$

Since we are conditioning on  $\bar{\Phi}$  occurring at time  $\tau$  there is a region of  $U \times \mathbb{R}^3$  that is ruled out for the possible initial position and velocity of the background particle - since we know that a background particle cannot have initial data that will lead it to collide with the tagged particle up to time  $\tau$ . This region is exactly where  $\mathbb{1}_\tau^\varepsilon[\Phi]$  equals zero. Hence for any  $(x_*, v_*) \in U \times \mathbb{R}^3$ ,

$$\hat{P}_\tau((x_1, v_1) = (x_*, v_*) | \bar{\Phi}) = \frac{g_0(v_*) \mathbb{1}_\tau^\varepsilon[\Phi](x_*, v_*)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}}.$$

Since  $\Phi \in \mathcal{G}(\varepsilon)$  we know that the background particle corresponding with the final collision, which has initial data  $(x(\tau) + \varepsilon\nu - \tau v', v')$ , does not collide with the tagged particle up to time  $\tau$ . Hence  $\mathbb{1}_\tau^\varepsilon[\Phi]$  is one at this point and (2.4.36) follows.

□

*Proof of theorem 2.4.6.* Combining lemmas 2.4.10, 2.4.17 and 2.4.19 gives the required proof of the theorem. □

## 2.5 Convergence

Having proven the existence of the idealised distribution  $P_t$  and shown that the empirical distribution  $\hat{P}_t$  solves the appropriate equation, we seek to show the convergence results that will help prove our main result. Following [38], the idea is to establish a differential inequality in (2.5.3). In combination with the fact that  $P_t$  is a probability measure and that  $\lim_{\varepsilon \rightarrow 0} P_t(\mathcal{G}(\varepsilon)) = 1$  in proposition 2.5.5 the inequality delivers the convergence result theorem 2.5.8. The main theorem 2.2.4 is a direct consequence. We first introduce some notation. Recall the definition of  $\mathbb{1}_t^\varepsilon[\Phi]$  and  $\zeta(\varepsilon)$  in (2.4.1) and (2.4.3) respectively. For  $\varepsilon > 0$ ,  $\Phi \in \mathcal{G}(\varepsilon)$ ,  $t \in [0, T]$ , define the following,

$$\begin{aligned} \eta_t^\varepsilon(\Phi) &:= \int_{U \times \mathbb{R}^3} g_0(\bar{v})(1 - \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v})) \, d\bar{x} \, d\bar{v}, \\ R_t^\varepsilon(\Phi) &:= \zeta(\varepsilon)P_t(\Phi), \\ L(\Phi) &:= - \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v})[(v(\tau) - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu, \\ C(\Phi) &:= 2 \sup_{t \in [0, T]} \left\{ \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v})[(v(t) - \bar{v}) \cdot \nu]_+ \right\} \\ \rho_t^{\varepsilon, 0}(\Phi) &:= \eta_t^\varepsilon(\Phi)C(\Phi)t. \end{aligned} \tag{2.5.1}$$

Further for  $k \geq 1$  define,

$$\rho_t^{\varepsilon, k}(\Phi) := (1 - \varepsilon)\rho_t^{\varepsilon, k-1}(\Phi) + \rho_t^{\varepsilon, 0}(\Phi) + \varepsilon.$$

and define,

$$\hat{\rho}_t^\varepsilon(\Phi) := \rho_t^{\varepsilon, n(\Phi)}(\Phi).$$

This recursive formula for  $\hat{\rho}_t^\varepsilon(\Phi)$  is used in the following proposition where we employ an inductive proof on the number of collisions in a history. Note that for  $k \geq 1$ ,

$$\rho_t^{\varepsilon, k}(\Phi) = (1 - \varepsilon)^k \rho_t^{\varepsilon, 0}(\Phi) + (\rho_t^{\varepsilon, 0}(\Phi) + \varepsilon) \sum_{j=1}^k (1 - \varepsilon)^{k-j}. \tag{2.5.2}$$

**Proposition 2.5.1.** *For  $\varepsilon$  sufficiently small,  $\Phi \in \mathcal{G}(\varepsilon)$  and  $t \in [0, T]$ ,*

$$\hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) \geq -\hat{\rho}_t^\varepsilon(\Phi) R_t^\varepsilon(\Phi). \quad (2.5.3)$$

To prove this proposition we use a number of lemmas.

**Lemma 2.5.2.** *For  $\Phi \in \mathcal{G}(\varepsilon)$  and  $t \geq \tau$ ,*

$$\begin{aligned} \hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) &\geq \exp\left(L(\Phi) \int_\tau^t (1 + 2\eta_s^\varepsilon(\Phi)) \, ds\right) (\hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi)) \\ &\quad + 2\eta_t^\varepsilon(\Phi) L(\Phi) R_t^\varepsilon(\Phi) \int_\tau^t \exp(2\eta_s^\varepsilon(\Phi)(t-s)L(\Phi)) \, ds. \end{aligned}$$

*Proof.* For  $t = \tau$  the result holds trivially. For  $t > \tau$ , by theorem 2.3.1 and theorem 2.4.6,

$$\partial_t \left( \hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) \right) = (1 - \gamma(t)) \hat{L}_t(\Phi) \hat{P}_t^\varepsilon(\Phi) - L(\Phi) R_t^\varepsilon(\Phi), \quad (2.5.4)$$

where

$$\hat{L}_t(\Phi) := - \left( \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(\tau) - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} \right).$$

For  $\varepsilon$  sufficiently small by (2.4.17),

$$\begin{aligned} \int_{U \times \mathbb{R}^3} g_0(\bar{v}) (1 - \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v})) \, d\bar{x} \, d\bar{v} &\leq \int_{\mathbb{R}^3} g_0(\bar{v}) \pi \varepsilon^2 \int_0^t |v(s) - \bar{v}| \, ds \, d\bar{v} \\ &\leq \int_{\mathbb{R}^3} g_0(\bar{v}) \pi \varepsilon^2 T(\mathcal{V}(\Phi) + |\bar{v}|) \, d\bar{v} \\ &\leq \pi \varepsilon^2 T \int_{\mathbb{R}^3} g_0(\bar{v}) (V(\varepsilon) + |\bar{v}|) \, d\bar{v} \leq \pi \varepsilon^2 T(V(\varepsilon) + \beta) < 1/2. \end{aligned} \quad (2.5.5)$$

Noting that for  $0 \leq z \leq 1/2$ ,  $\sum_{i=0}^\infty z^i \leq 2$  which gives,

$$\frac{1}{1-z} = \sum_{i=0}^\infty z^i = 1 + z \left( \sum_{i=0}^\infty z^i \right) \leq 1 + 2z.$$

It follows,

$$\begin{aligned} \frac{1}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} &= \frac{1}{1 - \int_{U \times \mathbb{R}^3} g_0(\bar{v}) (1 - \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v})) \, d\bar{x} \, d\bar{v}} \\ &\leq 1 + 2 \left( \int_{U \times \mathbb{R}^3} g_0(\bar{v}) (1 - \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v})) \, d\bar{x} \, d\bar{v} \right) = 1 + 2\eta_t^\varepsilon(\Phi). \end{aligned} \quad (2.5.6)$$

This gives that,

$$\begin{aligned}
(1 - \gamma(t)) & \left( \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{I}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} \right) \\
& \leq \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{I}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} \\
& \leq (1 + 2\eta_t^\varepsilon(\Phi)) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu.
\end{aligned}$$

Finally giving that,

$$(1 - \gamma(t)) \hat{L}_t(\Phi) \geq (1 + 2\eta_t^\varepsilon(\Phi)) L(\Phi).$$

Substituting this into (2.5.4),

$$\begin{aligned}
\partial_t \left( \hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) \right) &= (1 - \gamma(t)) \hat{L}_t(\Phi) \hat{P}_t(\Phi) - L(\Phi) R_t^\varepsilon(\Phi) \\
&\geq (1 + 2\eta_t^\varepsilon(\Phi)) L(\Phi) \hat{P}_t(\Phi) - L(\Phi) R_t^\varepsilon(\Phi) \\
&= (1 + 2\eta_t^\varepsilon(\Phi)) L(\Phi) \left( \hat{P}_t(\Phi) - R_t^\varepsilon(\Phi) \right) + 2\eta_t^\varepsilon(\Phi) L(\Phi) R_t^\varepsilon(\Phi).
\end{aligned} \tag{2.5.7}$$

For fixed  $\Phi$  this is simply a 1d ODE in  $t$ . If  $y : [\tau, \infty) \rightarrow \mathbb{R}$  satisfies,

$$\begin{cases} \frac{d}{dt} y(t) \geq a(t)y(t) + b(t), \\ y(\tau) = y_0. \end{cases}$$

Then it follows by the variation of constants,

$$y(t) \geq \exp \left( \int_\tau^t a(s) ds \right) y_0 + \int_\tau^t \exp \left( \int_s^t a(\sigma) d\sigma \right) b(s) ds.$$

Applying this to (2.5.7),

$$\begin{aligned}
\hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) &\geq \exp \left( \int_\tau^t (1 + 2\eta_s^\varepsilon(\Phi)) L(\Phi) ds \right) (\hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi)) \\
&\quad + \int_\tau^t \exp \left( \int_s^t (1 + 2\eta_\sigma^\varepsilon(\Phi)) L(\Phi) d\sigma \right) 2\eta_s^\varepsilon(\Phi) L(\Phi) R_s^\varepsilon(\Phi) ds.
\end{aligned} \tag{2.5.8}$$

Recall the definition of  $\mathbb{I}_t^\varepsilon[\Phi]$  (2.4.1) and note that it is non-increasing in  $t$ . Hence  $\eta_t^\varepsilon(\Phi)$  is non-decreasing. So for  $\tau \leq \sigma \leq t$ ,  $\eta_\sigma^\varepsilon(\Phi) \leq \eta_t^\varepsilon(\Phi)$ . Recalling that  $L(\Phi)$  is

non-positive, (2.5.8) becomes,

$$\begin{aligned}
\hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) &\geq \exp\left(\int_\tau^t (1 + 2\eta_s^\varepsilon(\Phi))L(\Phi) \, ds\right) (\hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi)) \\
&\quad + 2\eta_t^\varepsilon(\Phi) \int_\tau^t \exp\left(\int_s^t (1 + 2\eta_\sigma^\varepsilon(\Phi))L(\Phi) \, d\sigma\right) L(\Phi) R_s^\varepsilon(\Phi) \, ds \\
&\geq \exp\left(\int_\tau^t (1 + 2\eta_s^\varepsilon(\Phi))L(\Phi) \, ds\right) (\hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi)) \\
&\quad + 2\eta_t^\varepsilon(\Phi) L(\Phi) \int_\tau^t \exp\left((1 + 2\eta_s^\varepsilon(\Phi))(t - s) L(\Phi)\right) R_s^\varepsilon(\Phi) \, ds.
\end{aligned} \tag{2.5.9}$$

Finally for  $t > \tau$ ,

$$\partial_t R_t^\varepsilon(\Phi) = \zeta(\varepsilon) \partial_t P_t(\Phi) = \zeta(\varepsilon) P_t(\Phi) L(\Phi) = R_t^\varepsilon(\Phi) L(\Phi).$$

Hence for  $\tau \leq s \leq t$ ,

$$R_t^\varepsilon(\Phi) = \exp((t - s)L(\Phi)) R_s^\varepsilon(\Phi).$$

Which implies,

$$R_s^\varepsilon(\Phi) = \exp(-(t - s)L(\Phi)) R_t^\varepsilon(\Phi). \tag{2.5.10}$$

Substituting this into (2.5.9),

$$\begin{aligned}
\hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) &\geq \exp\left(\int_\tau^t (1 + 2\eta_s^\varepsilon(\Phi))L(\Phi) \, ds\right) (\hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi)) \\
&\quad + 2\eta_t^\varepsilon(\Phi) L(\Phi) \int_\tau^t \exp\left((1 + 2\eta_s^\varepsilon(\Phi))(t - s)L(\Phi)\right) \\
&\quad \times \exp(-(t - s)L(\Phi)) R_t^\varepsilon(\Phi) \, ds \\
&\geq \exp\left(\int_\tau^t (1 + 2\eta_s^\varepsilon(\Phi))L(\Phi) \, ds\right) (\hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi)) \\
&\quad + 2\eta_t^\varepsilon(\Phi) L(\Phi) R_t^\varepsilon(\Phi) \int_\tau^t \exp(2\eta_s^\varepsilon(\Phi)(t - s)L(\Phi)) \, ds.
\end{aligned}$$

This completes the proof of the lemma. □

**Lemma 2.5.3.** For  $\Phi \in \mathcal{G}(\varepsilon)$  and  $t \geq \tau$ ,

$$2\eta_t^\varepsilon(\Phi) L(\Phi) \int_\tau^t \exp(2\eta_s^\varepsilon(\Phi)(t - s)L(\Phi)) \, ds \geq -\rho_t^{\varepsilon,0}(\Phi).$$

*Proof.* Since  $L(\Phi) \leq 0$ ,

$$\int_{\tau}^t \exp(2\eta_s^{\varepsilon}(\Phi)(t-s)L(\Phi)) \, ds \leq t - \tau \leq t.$$

Recalling (2.5.1),

$$-L(\Phi) \leq \frac{C(\Phi)}{2}.$$

Hence combining these,

$$2L(\Phi) \int_{\tau}^t \exp(2\eta_s^{\varepsilon}(\Phi)(t-s)L(\Phi)) \, ds \geq -C(\Phi)t.$$

Multiplying both sides by  $\eta_t^{\varepsilon}(\Phi)$  gives the required identity.  $\square$

**Lemma 2.5.4.** *For  $\varepsilon$  sufficiently small, for any  $\Phi \in \mathcal{G}(\varepsilon)$  and any  $t \in [0, T]$ ,*

$$1 - \frac{1 - \gamma(t)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{I}_{\tau}^{\varepsilon}[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} \leq \varepsilon.$$

*Proof.* We show that for  $\varepsilon$  sufficiently small,

$$\frac{1}{\varepsilon} \left( 1 - \frac{1 - \gamma(t)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{I}_{\tau}^{\varepsilon}[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} \right) \leq 1. \quad (2.5.11)$$

Firstly by (2.5.5),

$$\int_{U \times \mathbb{R}^3} g_0(\bar{v}) (1 - \mathbb{I}_t^{\varepsilon}[\Phi](\bar{x}, \bar{v})) \, d\bar{x} \, d\bar{v} \leq \pi \varepsilon^2 T (V(\varepsilon) + \beta).$$

By (2.4.10) this converges to zero as  $\varepsilon$  converges to zero. Hence,

$$\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{I}_t^{\varepsilon}[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v} = 1 - \int_{U \times \mathbb{R}^3} g_0(\bar{v}) (1 - \mathbb{I}_t^{\varepsilon}[\Phi](\bar{x}, \bar{v})) \, d\bar{x} \, d\bar{v}$$

converges to one as  $\varepsilon$  converges to zero. It follows that,

$$\begin{aligned}
& \frac{1}{\varepsilon} \left( 1 - \frac{1 - \gamma^\varepsilon(t)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} \right) \\
&= \frac{1}{\varepsilon} \left( \frac{1 - \int_{U \times \mathbb{R}^3} g_0(\bar{v}) (1 - \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v})) \, d\bar{x} \, d\bar{v}}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} - \frac{1 - \gamma^\varepsilon(t)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} \right) \\
&\leq \frac{1}{\varepsilon} \left( \frac{\gamma^\varepsilon(t)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} \right) \\
&\leq \frac{\varepsilon n(\Phi)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} \leq \frac{\varepsilon M(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}}.
\end{aligned}$$

By (2.4.11) the numerator converges to zero as  $\varepsilon$  converges to zero, and hence, since the denominator converges to one, the expression converges to zero. Thus (2.5.11) holds for  $\varepsilon$  sufficiently small.  $\square$

*Proof of proposition 2.5.1.* We prove by induction on the number of collisions in  $\Phi$ . Firstly we show that the proposition holds for  $\Phi \in \mathcal{T}_0 \cap \mathcal{G}(\varepsilon)$ . Now if  $\Phi \in \mathcal{T}_0 \cap \mathcal{G}(\varepsilon)$  it follows that  $\tau = 0$  and hence,

$$\hat{P}_\tau^\varepsilon(\Phi) = \hat{P}_0^\varepsilon(\Phi) = \zeta(\varepsilon) f_0(x_0, v_0) = \zeta(\varepsilon) P_0(\Phi) = R_\tau^\varepsilon(\Phi).$$

By lemma 2.5.2 and 2.5.3 for  $t \geq 0$ ,

$$\begin{aligned}
\hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) &\geq 2\eta_t^\varepsilon(\Phi) L(\Phi) R_t^\varepsilon(\Phi) \int_\tau^t \exp(2\eta_s^\varepsilon(\Phi)(t-s)L(\Phi)) \, ds \geq -\rho_t^{\varepsilon,0}(\Phi) R_t^\varepsilon(\Phi) \\
&= -\hat{\rho}_t^\varepsilon(\Phi) R_t^\varepsilon(\Phi).
\end{aligned}$$

Proving the proposition in the base case. Now suppose that the proposition holds true for all histories  $\Phi \in \mathcal{T}_{k-1} \cap \mathcal{G}(\varepsilon)$  for some  $k \geq 1$  and let  $\Psi \in \mathcal{T}_k \cap \mathcal{G}(\varepsilon)$ . For  $t < \tau$  the proposition holds trivially, so we consider  $t \geq \tau$ . By theorem 2.4.6,

$$\hat{P}_\tau^\varepsilon(\Psi) = \frac{1 - \gamma(t)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} \hat{P}_\tau^\varepsilon(\bar{\Psi}) g_0(v') [(v(\tau^-) - v') \cdot \nu]_+,$$

and by theorem 2.3.1,

$$R_\tau^\varepsilon(\Psi) = R_\tau^\varepsilon(\bar{\Psi}) g_0(v') [(v(\tau^-) - v') \cdot \nu]_+.$$

Further since  $\bar{\Psi} \in \mathcal{T}_{k-1}$  we know by our inductive assumption that the proposition



holds for  $\bar{\Psi}$ , which implies.

$$\hat{P}_t^\varepsilon(\bar{\Psi}) \geq R_t^\varepsilon(\bar{\Psi}) - \hat{\rho}_t^\varepsilon(\bar{\Psi})R_t^\varepsilon(\bar{\Psi}).$$

Hence by the estimate in lemma 2.5.4 for  $\varepsilon$  sufficiently small,

$$\begin{aligned} \hat{P}_\tau^\varepsilon(\Psi) - R_\tau^\varepsilon(\Psi) &= g_0(v')[(v(\tau^-) - v') \cdot \nu]_+ \left( \frac{1 - \gamma(t)}{\int_{U \times \mathbb{R}^3} g_0(\bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} \hat{P}_\tau^\varepsilon(\bar{\Psi}) - R_\tau^\varepsilon(\bar{\Psi}) \right) \\ &\geq g_0(v')[(v(\tau^-) - v') \cdot \nu]_+ \left( (1 - \varepsilon) \hat{P}_\tau^\varepsilon(\bar{\Psi}) - R_\tau^\varepsilon(\bar{\Psi}) \right) \\ &\geq g_0(v')[(v(\tau^-) - v') \cdot \nu]_+ \left( (1 - \varepsilon) (R_\tau^\varepsilon(\bar{\Psi}) - \hat{\rho}_\tau^\varepsilon(\bar{\Psi}) R_\tau^\varepsilon(\bar{\Psi})) - R_\tau^\varepsilon(\bar{\Psi}) \right) \\ &= g_0(v')[(v(\tau^-) - v') \cdot \nu]_+ R_\tau^\varepsilon(\bar{\Psi}) (1 - \varepsilon - (1 - \varepsilon) \hat{\rho}_\tau^\varepsilon(\bar{\Psi}) - 1) \\ &= g_0(v')[(v(\tau^-) - v') \cdot \nu]_+ R_\tau^\varepsilon(\bar{\Psi}) (-\varepsilon - (1 - \varepsilon) \hat{\rho}_\tau^\varepsilon(\bar{\Psi})) \\ &= R_\tau^\varepsilon(\Psi) (-\varepsilon - (1 - \varepsilon) \hat{\rho}_\tau^\varepsilon(\bar{\Psi})). \end{aligned} \tag{2.5.12}$$

Since the trajectory of the tagged particle of  $\Psi$  up to time  $\tau$  is equal to the trajectory of the tagged particle of  $\bar{\Psi}$  up to time  $\tau$  and recalling that for any  $\Phi$ ,  $\eta_t^\varepsilon(\Phi)$  is non-decreasing in  $t$ , it follows,

$$\begin{aligned} \eta_\tau^\varepsilon(\bar{\Psi}) &= \int_{U \times \mathbb{R}^3} g_0(\bar{v}) (1 - \mathbb{1}_\tau^\varepsilon[\bar{\Psi}](\bar{x}, \bar{v})) d\bar{x} d\bar{v} \\ &= \int_{U \times \mathbb{R}^3} g_0(\bar{v}) (1 - \mathbb{1}_\tau^\varepsilon[\Psi](\bar{x}, \bar{v})) d\bar{x} d\bar{v} = \eta_\tau^\varepsilon(\Psi) \leq \eta_t^\varepsilon(\Psi), \end{aligned}$$

and recalling (2.5.1),  $C(\bar{\Psi}) \leq C(\Psi)$ . Hence,

$$\rho_\tau^{\varepsilon,0}(\bar{\Psi}) = \eta_\tau^\varepsilon(\bar{\Psi}) C(\bar{\Psi}) \tau \leq \eta_t^\varepsilon(\Psi) C(\Psi) t = \rho_t^{\varepsilon,0}(\Psi).$$

Recalling (2.5.2), this implies,

$$\hat{\rho}_\tau^\varepsilon(\bar{\Psi}) = \rho_\tau^{\varepsilon,k-1}(\bar{\Psi}) \leq \rho_t^{\varepsilon,k-1}(\Psi).$$

Substituting this into (2.5.12),

$$\begin{aligned} \hat{P}_\tau^\varepsilon(\Psi) - R_\tau^\varepsilon(\Psi) &\geq R_\tau^\varepsilon(\Psi) (-\varepsilon - (1 - \varepsilon) \hat{\rho}_\tau^\varepsilon(\bar{\Psi})) \\ &\geq R_\tau^\varepsilon(\Psi) (-\varepsilon - (1 - \varepsilon) \rho_t^{\varepsilon,k-1}(\Psi)) \\ &= -R_\tau^\varepsilon(\Psi) (\varepsilon + (1 - \varepsilon) \rho_t^{\varepsilon,k-1}(\Psi)). \end{aligned}$$

Recalling (2.5.10),

$$\begin{aligned}
& \exp \left( L(\Psi) \int_{\tau}^t (1 + 2\eta_s^{\varepsilon}(\Psi)) \, ds \right) (\hat{P}_{\tau}^{\varepsilon}(\Psi) - R_{\tau}^{\varepsilon}(\Psi)) \\
& \geq -\exp \left( L(\Psi) \int_{\tau}^t (1 + 2\eta_s^{\varepsilon}(\Psi)) \, ds \right) R_{\tau}^{\varepsilon}(\Psi) \left( \varepsilon + (1 - \varepsilon)\rho_t^{\varepsilon, k-1}(\Psi) \right) \\
& = -\exp \left( L(\Psi) \int_{\tau}^t 2\eta_s^{\varepsilon}(\Psi) \, ds \right) R_t^{\varepsilon}(\Psi) \left( \varepsilon + (1 - \varepsilon)\rho_t^{\varepsilon, k-1}(\Psi) \right) \\
& \geq -R_t^{\varepsilon}(\Psi) \left( \varepsilon + (1 - \varepsilon)\rho_t^{\varepsilon, k-1}(\Psi) \right).
\end{aligned}$$

Recalling lemma 2.5.2 and 2.5.3 this gives,

$$\begin{aligned}
\hat{P}_t^{\varepsilon}(\Psi) - R_t^{\varepsilon}(\Psi) & \geq \exp \left( \int_{\tau}^t L(\Psi)(1 + 2\eta_s^{\varepsilon}(\Psi)) \, ds \right) (\hat{P}_{\tau}^{\varepsilon}(\Psi) - R_{\tau}^{\varepsilon}(\Psi)) \\
& \quad + 2\eta_t^{\varepsilon}(\Psi)L(\Psi)R_t^{\varepsilon}(\Psi) \int_{\tau}^t \exp(2\eta_s^{\varepsilon}(\Psi)(t-s)L(\Psi)) \, ds \\
& \geq -R_t^{\varepsilon}(\Psi) \left( \varepsilon + (1 - \varepsilon)\rho_t^{\varepsilon, k-1}(\Psi) \right) - \rho_t^{\varepsilon, 0}(\Psi)R_t^{\varepsilon}(\Psi) \\
& \geq -R_t^{\varepsilon}(\Psi) \left( \varepsilon + (1 - \varepsilon)\rho_t^{\varepsilon, k-1}(\Psi) + \rho_t^{\varepsilon, 0}(\Psi) \right) \\
& = -R_t^{\varepsilon}(\Psi)\rho_t^{\varepsilon, k}(\Psi) = -R_t^{\varepsilon}(\Psi)\hat{\rho}_t^{\varepsilon}(\Psi).
\end{aligned}$$

This completes the proof of the inductive step which concludes the proof of the proposition.  $\square$

**Proposition 2.5.5.** *Good histories have full measure in the sense that*

$$\lim_{\varepsilon \rightarrow 0} P_t(\mathcal{MT} \setminus \mathcal{G}(\varepsilon)) = 0.$$

*Proof.* Firstly we prove that  $\mathcal{G}(0)$  is of measure 1. To this aim note that  $\mathcal{T}_0 \setminus R(0)$  is empty because a history in  $\mathcal{T}_0$  cannot include a re-collision. Now let  $\Phi \in \mathcal{T}_1$  and denote  $\Phi = ((x_0, v_0), (\tau, \nu, v'))$ . If  $\Phi \in \mathcal{T}_1 \setminus R(0)$  then there exists an  $s \in (\tau, T]$  and an  $m \in \mathbb{Z}^3$  such that  $x(s) + m = x_1(s)$ . We show that this forces  $\nu$  to be in a set of zero measure. Note that  $x(s) = x_0 + \tau v_0 + (s - \tau)v(\tau)$ , where  $v(\tau) = v_0 - \nu(v_0 - v') \cdot \nu$ , and  $x_1(s) = x_0 + \tau v_0 - \tau v' + s v'$ . Hence,

$$x_0 + \tau v_0 + (s - \tau)v(\tau) + m = x_0 + \tau v_0 - \tau v' + s v',$$

which implies  $m = (s - \tau)(v' - v(\tau))$ . Hence,

$$m \cdot \nu = (s - \tau)(v' - v(\tau)) \cdot \nu = (s - \tau)(v' \cdot \nu - (v_0 - \nu(v_0 - v') \cdot \nu)) \cdot \nu = 0.$$

That is, if  $\Phi \in \mathcal{T}_1 \setminus R(0)$  then there exists an  $m \in \mathbb{Z}^3$  such that  $m \cdot \nu = 0$ . Hence in  $\nu$  must be in a set of zero measure. Thus  $\mathcal{T}_1 \setminus R(0)$  is a set of zero measure.

Now let  $j \geq 2$  and  $\Phi \in \mathcal{T}_j$  with  $\Phi = ((x_0, v_0), (t_1, \nu_1, v_1), \dots, (t_j, \nu_j, v_j))$ . If  $\Phi \in \mathcal{T}_j \setminus R(0)$  then either two of the collisions correspond to the same background particle or the tagged particle will re-collide with one of the particles at some time  $s \in (\tau, T]$ . In the first case this implies there exists  $2 \leq l \leq j$  and  $1 \leq k < l$  such that the  $k$ th and  $l$ th collision corresponds to the same background particle. Hence  $v_l = v_k$  and thus  $v_l$  is restricted to a set of zero measure.

In the second case there exists an  $s \in (\tau, T]$ ,  $m \in \mathbb{Z}^3$  and  $1 \leq k \leq j$  such that  $x(s) + m = x_k(s)$ . But,

$$x(s) = x_0 + t_1 v(t_0) + (t_2 - t_1) v(t_2) + \dots + (t_j - t_{j-1}) v(t_{j-1}) + (s - t_j) v(t_j),$$

and,

$$\begin{aligned} x_k(s) &= x_k(t_k) + (s - t_k) v_k = x(t_k) + (s - t_k) v_k \\ &= x_0 + t_1 v_0 + (t_2 - t_1) v(t_2) + \dots + (t_k - t_{k-1}) v(t_{k-1}) + (s - t_k) v_k. \end{aligned}$$

Hence,

$$(t_k - t_{k-1}) v(t_{k-1}) + \dots + (t_j - t_{j-1}) v(t_{j-1}) + (s - t_j) v(t_j) + m = (s - t_k) v_k. \quad (2.5.13)$$

Now  $v(t_j) = v(t_{j-1}) - \nu_j(v(t_{j-1}) - v_j) \cdot \nu_j$ , giving  $v(t_j) \cdot \nu_j = v_j \cdot \nu_j$ . Hence taking the dot product of (2.5.13) with  $\nu_j$  gives,

$$((t_k - t_{k-1}) v(t_{k-1}) + \dots + (t_j - t_{j-1}) v(t_{j-1})) \cdot \nu_j + m \cdot \nu_j - (s - t_k) v_k \cdot \nu_j = -(s - t_j) v_j \cdot \nu_j.$$

For all components of  $\Phi$  given apart from  $v_j$  this implies that  $v_j$  must be in a set of zero measure. Hence  $\mathcal{T}_j \setminus R(0)$  is a set of zero measure. It follows that

$$\mathcal{MT} \setminus R(0) = \cup_{j \geq 0} \mathcal{T}_j \setminus R(0)$$

is also a set of zero measure. Therefore, since the other requirements on  $\mathcal{G}(0)$  are clear,  $P_t(\mathcal{MT} \setminus \mathcal{G}(0)) = 0$ , and hence also that  $P_t(\mathcal{G}(0)) = 1$ .

Since  $\mathcal{G}(\varepsilon)$  is increasing as  $\varepsilon$  decreases and  $\lim_{\varepsilon \rightarrow 0} \mathcal{G}(\varepsilon) = \mathcal{G}(0)$  it follows by the dominated convergence theorem that,

$$\lim_{\varepsilon \rightarrow 0} P_t(\mathcal{G}(\varepsilon)) = P_t(\mathcal{G}(0)) = 1.$$

Hence

$$\lim_{\varepsilon \rightarrow 0} P_t(\mathcal{MT} \setminus \mathcal{G}(\varepsilon)) = 0,$$

as required.  $\square$

**Lemma 2.5.6.** *Recall  $\beta$  (2.4.17). For  $\varepsilon > 0$ ,  $\Phi \in \mathcal{G}(\varepsilon)$  there exists constants  $C_1, C_2 > 0$  such that,*

$$\eta_t^\varepsilon(\Phi) \leq C_1 \varepsilon^2 (\beta + V(\varepsilon)), \quad (2.5.14)$$

$$C(\Phi) \leq C_2 (\beta + V(\varepsilon)). \quad (2.5.15)$$

*Proof.* Firstly by (2.5.5),

$$\eta_t^\varepsilon(\Phi) = \int_{U \times \mathbb{R}^3} g_0(\bar{v}) (1 - \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v})) \, d\bar{x} \, d\bar{v} \leq \pi \varepsilon^2 T (\beta + V(\varepsilon)).$$

So take  $C_1 := \pi T$  proving (2.5.14). Next note,

$$\begin{aligned} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_0(\bar{v}) [(v(t) - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu &\leq \pi \int_{\mathbb{R}^3} g_0(\bar{v}) |v(t) - \bar{v}| \, d\bar{v} \\ &\leq \pi \int_{\mathbb{R}^3} g_0(\bar{v}) (\mathcal{V}(\Phi) + |\bar{v}|) \, d\bar{v} \leq \pi (V(\varepsilon) + \beta). \end{aligned}$$

Hence by (2.5.1),

$$C(\Phi) \leq 2\pi (V(\varepsilon) + \beta).$$

so take  $C_2 := 2\pi$  which proves (2.5.15).  $\square$

**Lemma 2.5.7.** *For any  $\delta > 0$ , there exists a  $\varepsilon' > 0$  such that for  $0 < \varepsilon < \varepsilon'$  and for any  $\Phi \in \mathcal{G}(\varepsilon)$ ,*

$$\hat{\rho}_t^\varepsilon(\Phi) < \delta.$$

*Proof.* Fix  $\delta > 0$ . Firstly by the above lemma,

$$\rho_t^{\varepsilon,0}(\Phi) = \eta_t(\Phi) C(\Phi) t \leq C_1 C_2 T \varepsilon^2 (\beta + V(\varepsilon))^2.$$

Recalling (2.4.10) there exists an  $\varepsilon_1 > 0$  such that for  $0 < \varepsilon < \varepsilon_1$ ,

$$\rho_t^{\varepsilon,0}(\Phi) < \frac{\delta}{3}.$$

Further there exists an  $\varepsilon_2 > 0$  such that for  $0 < \varepsilon < \varepsilon_2$ ,

$$\frac{1}{\sqrt{\varepsilon}}\rho_t^{\varepsilon,0}(\Phi) \leq C_1 C_2 T \varepsilon^{3/2} (\beta + V(\varepsilon))^2 < \frac{\delta}{3}.$$

Finally there exists an  $\varepsilon_3 > 0$  such that for  $0 < \varepsilon < \varepsilon_3$ ,  $\sqrt{\varepsilon} < \frac{\delta}{3}$ . Hence take  $\varepsilon' = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, 1\}$  then for any  $\Phi \in \mathcal{G}(\varepsilon)$ ,

$$\begin{aligned} \hat{\rho}_t^\varepsilon(\Phi) &= \rho_t^{\varepsilon, n(\Phi)}(\Phi) = (1 - \varepsilon)^{n(\Phi)} \rho_t^{\varepsilon, 0}(\Phi) + (\rho_t^{\varepsilon, 0}(\Phi) + \varepsilon) \sum_{j=1}^{n(\Phi)} (1 - \varepsilon)^{n(\Phi)-j} \\ &\leq \rho_t^{\varepsilon, 0}(\Phi) + (\rho_t^{\varepsilon, 0}(\Phi) + \varepsilon) \times n(\Phi) \leq \rho_t^{\varepsilon, 0}(\Phi) + M(\varepsilon)(\rho_t^{\varepsilon, 0}(\Phi) + \varepsilon) \\ &= \rho_t^{\varepsilon, 0}(\Phi) + M(\varepsilon)\rho_t^{\varepsilon, 0}(\Phi) + M(\varepsilon)\varepsilon \leq \rho_t^{\varepsilon, 0}(\Phi) + \frac{1}{\sqrt{\varepsilon}}\rho_t^{\varepsilon, 0}(\Phi) + \sqrt{\varepsilon} < \delta. \end{aligned}$$

□

**Theorem 2.5.8.** *Uniformly for  $t \in [0, T]$ ,*

$$\lim_{\varepsilon \rightarrow 0} \|P_t - \hat{P}_t^\varepsilon\|_{TV} = 0.$$

*Proof.* Let  $\delta > 0$  and  $S \subset \mathcal{MT}$  then,

$$\begin{aligned} P_t(S) - \hat{P}_t^\varepsilon(S) &= P_t(S \cap \mathcal{G}(\varepsilon)) + P_t(S \setminus \mathcal{G}(\varepsilon)) - \hat{P}_t^\varepsilon(S \cap \mathcal{G}(\varepsilon)) - \hat{P}_t^\varepsilon(S \setminus \mathcal{G}(\varepsilon)) \\ &\leq P_t(S \cap \mathcal{G}(\varepsilon)) + P_t(S \setminus \mathcal{G}(\varepsilon)) - \hat{P}_t^\varepsilon(S \cap \mathcal{G}(\varepsilon)). \end{aligned}$$

By proposition 2.5.5 for  $\varepsilon$  sufficiently small,

$$P_t(S \setminus \mathcal{G}(\varepsilon)) \leq P_t(\mathcal{MT} \setminus \mathcal{G}(\varepsilon)) < \frac{\delta}{3}.$$

Hence,

$$P_t(S) - \hat{P}_t^\varepsilon(S) < P_t(S \cap \mathcal{G}(\varepsilon)) - \hat{P}_t^\varepsilon(S \cap \mathcal{G}(\varepsilon)) + \frac{\delta}{3}. \quad (2.5.16)$$

Recall the definition of  $\zeta(\varepsilon)$  in (2.4.3). It is clear that this implies

$$\zeta(\varepsilon) \leq 1. \quad (2.5.17)$$

Hence by the above lemma, for  $\varepsilon$  sufficiently small and  $\Phi \in \mathcal{G}(\varepsilon)$ ,

$$\zeta(\varepsilon)\hat{\rho}_t^\varepsilon(\Phi) \leq \hat{\rho}_t^\varepsilon(\Phi) < \frac{\delta}{3}.$$

Further by proposition 2.5.1, for  $\Phi \in \mathcal{G}(\varepsilon)$ ,

$$-\hat{P}_t^\varepsilon(\Phi) \leq -R_t^\varepsilon(\Phi) + \hat{\rho}_t^\varepsilon(\Phi)R_t^\varepsilon(\Phi). \quad (2.5.18)$$

The Binomial inequality states that for  $x \geq -1$  and  $N \in \mathbb{N}$ ,

$$(1+x)^N \geq 1+Nx.$$

Hence for  $\varepsilon > 0$  such that  $\frac{4}{3}\pi\varepsilon^3 \leq 1$  we apply this to  $\zeta(\varepsilon)$  recalling (2.2.1),

$$\zeta(\varepsilon) = (1 - \frac{4}{3}\pi\varepsilon^3)^N \geq 1 - N\frac{4}{3}\pi\varepsilon^3 = 1 - \frac{4}{3}\pi\varepsilon. \quad (2.5.19)$$

Hence for  $\varepsilon$  sufficiently small (2.5.18) gives,

$$\begin{aligned} P_t(\Phi) - \hat{P}_t^\varepsilon(\Phi) &\leq P_t(\Phi) - R_t^\varepsilon(\Phi) + \hat{\rho}_t^\varepsilon(\Phi)R_t^\varepsilon(\Phi) \\ &= P_t(\Phi) - \zeta(\varepsilon)P_t(\Phi) + \hat{\rho}_t^\varepsilon(\Phi)\zeta(\varepsilon)P_t(\Phi) \\ &\leq \frac{4}{3}\pi\varepsilon P_t(\Phi) + \frac{\delta}{3}P_t(\Phi) \leq \frac{2\delta}{3}P_t(\Phi). \end{aligned}$$

This holds for all  $\Phi \in \mathcal{G}(\varepsilon)$  with  $\varepsilon$  sufficiently small hence

$$P_t(S \cap \mathcal{G}(\varepsilon)) - \hat{P}_t^\varepsilon(S \cap \mathcal{G}(\varepsilon)) \leq \frac{2\delta}{3}P_t(S \cap \mathcal{G}(\varepsilon)) \leq \frac{2\delta}{3}. \quad (2.5.20)$$

By substituting (2.5.20) into (2.5.16), for  $\varepsilon$  sufficiently small,

$$P_t(S) - \hat{P}_t^\varepsilon(S) < \delta. \quad (2.5.21)$$

Since  $\varepsilon$  did not depend on  $S$  this holds true for every  $S \subset \mathcal{MT}$ . Hence for any  $S \subset \mathcal{MT}$ ,

$$\begin{aligned} \hat{P}_t^\varepsilon(S) - P_t(S) &= (1 - \hat{P}_t^\varepsilon(\mathcal{MT} \setminus S)) - (1 - P_t(\mathcal{MT} \setminus S)) \\ &= P_t(\mathcal{MT} \setminus S) - \hat{P}_t^\varepsilon(\mathcal{MT} \setminus S) < \delta. \end{aligned}$$

This together with (2.5.21) gives that, for  $\varepsilon$  sufficiently small, for any  $S \subset \mathcal{MT}$ ,

$$|P_t(S) - \hat{P}_t^\varepsilon(S)| < \delta,$$

which completes the proof of the theorem.  $\square$

We can now prove the main theorem of this chapter, theorem 2.2.4, which follows from the above theorem.

*Proof of theorem 2.2.4.* Recall for  $\Omega \subset U \times \mathbb{R}^3$ ,

$$S_t(\Omega) := \{\Phi \in \mathcal{MT} : (x(t), v(t)) \in \Omega\}.$$

By theorem 2.3.1,

$$\int_{\Omega} f_t(x, v) \, dx \, dv = \int_{S_t(\Omega)} P_t(\Phi) \, d\Phi = P_t(S_t(\Omega)).$$

Also by definition of  $\hat{P}_t^\varepsilon$ ,

$$\int_{\Omega} \hat{f}_t^N(x, v) \, dx \, dv = \int_{S_t(\Omega)} \hat{P}_t^\varepsilon(\Phi) \, d\Phi = \hat{P}_t^\varepsilon(S_t(\Omega)).$$

Hence by theorem 2.5.8,

$$\begin{aligned} \lim_{N \rightarrow \infty} \sup_{\Omega \subset U \times \mathbb{R}^3} \left| \int_{\Omega} \hat{f}_t^N(x, v) - f_t(x, v) \, dx \, dv \right| \\ = \lim_{\varepsilon \rightarrow 0} \sup_{\Omega \subset U \times \mathbb{R}^3} |P_t(S_t(\Omega)) - \hat{P}_t^\varepsilon(S_t(\Omega))| = 0, \end{aligned}$$

which completes the proof. □

## 2.6 Proof of Auxiliary Results

### 2.6.1 Particle dynamics

*Proof of Prop 2.2.1.* The dynamics become undefined if there is instantaneously more than one background particle colliding with the tagged particle or if the tagged particle experiences an infinite number of collisions in finite time. We adapt a similar proof for the full hard-spheres dynamics from [20, Prop 4.1.1].

Let  $R > 0$  and  $\delta < \varepsilon/2$  such that there exists a  $K \in \mathbb{N}$  with  $T = K\delta$ . Denote the ball of radius  $R$  about  $x$  in  $\mathbb{R}^3$  by  $B_R(x)$ . For the initial position of the tagged particle  $(x_0, v_0) \in U \times B_R(0)$  fixed define  $I(x_0, v_0) \subset (U \times \mathbb{R}^3)^N$  by,

$$\begin{aligned} I(x_0, v_0) := \{(x_1, v_1), \dots, (x_N, v_N) \in (U \times B_R(0))^N : \text{the tagged particle collides} \\ \text{with at least two background particles in the time interval } [0, \delta]\}. \end{aligned}$$

We bound the volume of this set. Firstly define

$$I^1(x_0, v_0) := \{(x_1, v_1) \in U \times B_R(0) : \varepsilon \leq |x_0 - x_1| \leq \varepsilon + 2R\delta\}.$$

It can be seen that for some  $C$ ,

$$|I^1(x_0, v_0)| \leq CR^3 \times (2R\delta)^3.$$

Since  $I(x_0, v_0)$  is a subset of,

$$\{(x_1, v_1), \dots, (x_N, v_N) \in (U \times B_R(0))^N : \exists 1 \leq i < j \leq N \text{ such that} \\ \varepsilon \leq |x_0 - x_i| \leq \varepsilon + 2R\delta \text{ and } \varepsilon \leq |x_0 - x_j| \leq \varepsilon + 2R\delta\}$$

The above estimate gives, for some constant  $C = C(N, \varepsilon)$ ,

$$|I(x_0, v_0)| \leq CR^{3(N-2)} \times (R^6 \delta^3)^2 \\ \leq CR^{3(N+2)} \delta^6.$$

Hence if we define,

$$I := \cup \{I(x_0, v_0) : (x_0, v_0) \in U \times B_R(0)\},$$

it follows,

$$|I| \leq CR^{3(N+3)} \delta^6.$$

Hence there exists a subset  $I_0(\delta, R)$  of measure at most  $CR^{3(N+3)} \delta^6$  such that for any initial configuration in  $(U \times B_R(0))^{N+1} \setminus I_0(\delta, R)$  the tagged particle experiences at most one collision in  $[0, \delta]$ .

Now consider the system at time  $\delta$ . Since all particles had initial velocity in  $B_R(0)$  and the tagged particle had at most one collision in time  $[0, \delta]$  the velocity of the tagged particle at time  $\delta$  is in  $B_{2R}(0)$ . By the same arguments above there exists a set  $I_1(\delta, R)$  of measure at most  $CR^{3(N+3)} \delta^6$  for some new constant  $C$  such that for any initial configuration in  $(U \times B_R(0))^{N+1} \setminus I_0(\delta, R) \cup I_1(\delta, R)$  the tagged particle experiences at most one collision in  $[0, \delta]$  and at most one collision in  $[\delta, 2\delta]$  and thus the dynamics are well defined up to  $2\delta$ .

Continue this process  $K$  times defining the set,

$$I(\delta, R) := \cup_{j=0}^{K-1} I_j(\delta, R),$$

which has measure at most  $CR^{3(N+3)} \delta^6$  for some new constant  $C$  and such that for any initial configuration in  $(U \times B_R(0))^{N+1} \setminus I(\delta, R)$  the tagged particle has at most one collision per time interval  $[j\delta, (j+1)\delta]$  and hence the dynamics are well defined up



to time  $T$ . Defining,

$$I(T, R) := \cap_{\delta > 0} I(\delta, R),$$

it follows  $I(T, R)$  is of measure zero and for any any initial configuration in  $(U \times B_R(0))^{N+1} \setminus I(T, R)$  the dynamics are well defined up to time  $T$ .

Finally take,

$$I := \cup_{R \in \mathbb{N}} I(T, R)$$

and note that  $I$  is a countable union of measure zero sets and for any initial configuration in  $(U \times \mathbb{R}^3) \setminus I$  the dynamics are well defined up to time  $T$ .  $\square$

## Chapter 3

# The Non-Autonomous Case

### 3.1 Introduction

In this chapter we derive a non-autonomous linear Boltzmann equation from the Boltzmann-Grad limit of a Rayleigh gas particle model, where one tagged particle evolves amongst a large number of non self interacting background particles.

In contrast to chapter 2 the initial distribution of the background particles is now spatially non-homogeneous and we assume that at a collision between the tagged particle and a background particle there is a full hard sphere collision in which both particles change direction.

The main result is theorem 3.2.4, where it is shown that the distribution of the tagged particle evolving among  $N$  background particles converges in total variation as  $N$  tends to infinity to the solution of the non-autonomous linear Boltzmann equation. The convergence holds for arbitrarily large times and with moderate moment assumptions on the initial data.

We follow the same method as chapter 2, which closely follows [38]. The idealised equation on collision histories is stated and semigroup methods are used to show that there exists a solution. Then it is shown that the distribution on collision histories described by the dynamics solves the empirical equation. In section 3.5 convergence is shown between the solutions of the empirical and idealised equations, which then leads to the proof of the main theorem.

The biggest difference to chapter 2 is in section 3.3 on the idealised equation. The introduction of a spatial dependence on the initial distribution of the background creates non-autonomous equations, which require evolution semigroup results to study. There is no specific evolution semigroup result for us to refer to, so our problem is viewed in the framework of general evolution semigroup theory, which creates a number of more

technical calculations.

The question of honesty of the semigroup solution of the non-autonomous linear Boltzmann equation is also more difficult than the autonomous case, since we were unable to directly verify honesty from existing results. Instead honesty of the solution is proven indirectly via the connection to the idealised equation.

The change in collisions, where a collision between the tagged particle and a background particle is now a full hard sphere collision, makes only a minimal difference.

### 3.2 Model and Main Result

We now give our Rayleigh gas particle model in detail. The model differs from the model in chapter 2 in two ways: i) we no longer assume that the initial distribution of the background particles is spatially homogeneous and ii) now when the tagged particle collides with a background particle the collision is treated as a full hard sphere collision and so the background particle changes velocity rather than continuing with the same pre-collision velocity.

Let  $U = \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$  be the flat three dimensional unit torus. Let  $N \in \mathbb{N}$ . One tagged particle evolves amongst  $N$  background particles. The tagged particle has random initial position and velocity given by  $f_0 \in L^1(U \times \mathbb{R}^3)$  and the  $N$  background particles have random and independent initial position and velocity given by  $g_0 \in L^1(U \times \mathbb{R}^3)$ . The tagged particle and background particles are modelled as spheres with unit mass and diameter  $\varepsilon > 0$  given by the Boltzmann-Grad scaling,  $N\varepsilon^2 = 1$ .

The tagged particle travels with constant velocity while it remains at least  $\varepsilon$  away from all background particles. Each background particles travels with constant velocity while it remains at least  $\varepsilon$  away from the tagged particle. Background particles do not effect each other and freely pass through each other. When the position of the tagged particle comes within  $\varepsilon$  of the the position of a background particle both particles instantaneously change velocity as described by Newtonian hard-sphere collisions. We describe this process explicitly.

Let the position and velocity of the tagged particle at time  $t \geq 0$  be denoted  $(x(t), v(t))$  and for  $1 \leq j \leq N$ , let the position and velocity of background particle  $j$  at time  $t$  be denoted  $(x_j(t), v_j(t))$ . Then for all  $t \geq 0$ ,

$$\frac{dx(t)}{dt} = v(t) \text{ and } \frac{dx_j(t)}{dt} = v_j(t).$$

If there exists a  $1 \leq j \leq N$  such that  $|x(0) - x_j(0)| \leq \varepsilon$  then we assume that the two particles pass through each other unaffected (indeed this is well defined since the

velocities are only equal with probability zero). That is, any initial overlap is ignored and not treated as a collision. Now let  $t > 0$ . If for all  $1 \leq j \leq N$ ,  $|x(t) - x_j(t)| > \varepsilon$  then,

$$\frac{dv(t)}{dt} = 0 \text{ and } \frac{dv_j(t)}{dt} = 0.$$

Else there exists a  $1 \leq j \leq N$  such that  $|x(t) - x_j(t)| = \varepsilon$  and both particles experience an instantaneous collision at time  $t$ . We denote by  $v(t^-)$  and  $v_j(t^-)$  the velocity of the tagged particle and background particle  $j$  instantaneously before the collision and define  $v(t)$  and  $v_j(t)$  to be the velocity of the tagged particle and background particle  $j$  instantaneously after the collision. Define the collision parameter  $\nu \in \mathbb{S}^2$ ,

$$\nu := \frac{x(t) - x_j(t)}{|x(t) - x_j(t)|}.$$

Then  $v(t)$  and  $v_j(t)$  are given by,

$$\begin{aligned} v(t) &= v(t^-) - \nu \cdot (v(t^-) - v_j(t^-)) \cdot \nu \\ v_j(t) &= v_j(t^-) + \nu \cdot (v(t^-) - v_j(t^-)) \cdot \nu. \end{aligned}$$

In contrast to chapter 2, these particle dynamics are fully reversible since no information is lost at collisions.

**Proposition 3.2.1.** *For  $N \in \mathbb{N}$  and  $T > 0$  fixed these dynamics are well defined up to time  $T$  for all initial configurations apart from a set of zero measure.*

*Proof.* The proof of this is unchanged from proposition 2.2.1, which is based upon [20, proposition 4.1.1].  $\square$

**Definition 3.2.2.** For  $t \geq 0$  and  $N \in \mathbb{N}$  let  $\hat{f}_t^N$  denote the distribution of the tagged particle at time  $t$  evolving via the Rayleigh gas dynamics described above amongst  $N$  background particles.

We are interested in the behaviour of  $\hat{f}_t^N$  as  $N$  increases to infinity, or equivalently as  $\varepsilon$  converges to zero. In the main theorem of this chapter, theorem 3.2.4, we show that for any fixed  $T > 0$  and under some assumptions on  $f_0$  and  $g_0$ ,  $\hat{f}_t^N$  converges to  $f_t^0$ , the solution of the non-autonomous linear Boltzmann equation, in total variation as  $N$  tends to infinity uniformly for any  $t \in [0, T]$ .

**Definition 3.2.3.** Let  $f_0, g_0 \in L^1(U \times \mathbb{R}^3)$  be probability densities. Then  $f_0$  is said to be *tagged-admissible* if

$$\int_{U \times \mathbb{R}^3} f_0(x, v)(1 + |v|^2) dx dv =: M_f < \infty. \quad (3.2.1)$$

Define  $\bar{g} : \mathbb{R}^3 \rightarrow \mathbb{R}$  by

$$\bar{g}(v) := \operatorname{ess\,sup}_{x \in U} g_0(x, v). \quad (3.2.2)$$

Then  $g_0$  is *background-admissible* if all of the following hold,

$$\int_{\mathbb{R}^3} \bar{g}(v)(1 + |v|^2) \, dv =: M_g < \infty, \quad (3.2.3)$$

$$\operatorname{ess\,sup}_{v \in \mathbb{R}^3} \bar{g}(v)(1 + |v|) =: M_\infty < \infty, \quad (3.2.4)$$

for almost all  $v \in \mathbb{R}^3$ ,  $g_0(\cdot, v) \in W^{1,1}(U)$  and,

$$\operatorname{ess\,sup}_{x \in U} \int_{\mathbb{R}^3} |\nabla_x g_0(x, v)|(1 + |v|) \, dv =: M_1 < \infty, \quad (3.2.5)$$

and there exists a  $M > 0$  and an  $0 < \alpha \leq 1$  such that for almost all  $v \in \mathbb{R}^3$  and for any  $x, y \in U$ ,

$$|g_0(x, v) - g_0(y, v)| < M|x - y|^\alpha. \quad (3.2.6)$$

We now state the relevant non-autonomous linear Boltzmann equation. Firstly for  $t \geq 0$  define the operators  $Q_t^{0,+}$  and  $Q_t^{0,-} : L^1(U \times \mathbb{R}^3) \rightarrow L^1(U \times \mathbb{R}^3)$  by,

$$Q_t^{0,+}[f](x, v) := \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} f(x, v') g_t(x, \bar{v}') [(v - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu, \quad (3.2.7)$$

and,

$$Q_t^{0,-}[f](x, v) := f(x, v) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu, \quad (3.2.8)$$

where we use the notation  $g_t(x, v) := g_0(x - tv, v)$  and where the pre-collision velocities,  $v'$  and  $\bar{v}'$ , are given by  $v' = v + \nu \cdot (\bar{v} - v)\nu$  and  $\bar{v}' = \bar{v} - \nu \cdot (\bar{v} - v)\nu$ . Further define  $Q_t^0 := Q_t^{0,+} - Q_t^{0,-}$ . The non-autonomous linear Boltzmann equation is given by,

$$\begin{cases} \partial_t f_t^0(x, v) &= -v \cdot \nabla_x f_t^0(x, v) + Q_t^0[f_t^0](x, v), \\ f_{t=0}^0(x, v) &= f_0(x, v). \end{cases} \quad (3.2.9)$$

We now state the main theorem.

**Theorem 3.2.4.** *Let  $0 < T < \infty$  and suppose that  $f_0$  and  $g_0$  are tagged and background admissible probability densities respectively. Then, uniformly for  $t \in [0, T]$ ,  $\hat{f}_t^N$ , the distribution of the tagged particle at time  $t$  among  $N$  background particles under the above particle dynamics, converges in total variation as  $N$  tends to infinity to  $f_t^0$ , a solution of the non-autonomous linear Boltzmann equation (3.2.9).*

### 3.2.1 Remarks

1. We prove the result in dimension 3. The result should also hold in the case  $d = 2$  or  $d \geq 4$  up to a change in moment assumptions and a change in geometric constants throughout.
2. With stronger moment assumptions on the initial distributions  $f_0$  and  $g_0$  it may be possible to calculate explicit convergence rates. In particular to show (3.3.8) we use the dominated convergence theorem which proves convergence without any explicit rate. With further assumptions on our initial data it may be possible to prove this via an alternative method.
3. One could attempt to adapt these methods to more involved particle models, such as the addition of an external force acting on the particles. In such a situation the relevant linear Boltzmann equation would include the additional force term. Also the distribution of the background particles at time  $t$  would include the effects of this force. This would add additional complications to the various bounds computed throughout.
4. Another way to adapt these methods to a more complex particle system would be to model the background particles such that a collision happens between background particle  $i$  and  $j$  if both particles have experienced less than  $k$  collisions. One could then attempt to let  $k$  tend to infinity, which would result in a full hard-sphere particle model. This model would involve extra complexities in marking each background particle with the number of collisions it has so far experienced.

### 3.2.2 Method of Proof

We follow the same method as in chapter 2, which is based on [38]. We consider two Kolmogorov equations on the set of all possible collision histories. Section 3.3 is mostly devoted to proving theorem 3.3.1, where we prove that there exists a solution to the idealised equation by an iterative construction process and then prove that a number of properties hold, including the connection to the solution of the linear Boltzmann equation. In this section we introduce a  $\varepsilon$  dependence in both the idealised equation and the linear Boltzmann equation to enable convergence proofs that follow later.

In section 3.4 we prove that the the distribution of all possible collision histories from our particle dynamics solves the empirical equation, at least for well controlled situations, which resembles the idealised equation. We do this by explicitly calculating the rate of change of the distribution on all possible collision histories.

Finally in section 3.5 we prove the main theorem of the chapter, theorem 3.2.4, by proving the convergence between the solutions of the idealised and empirical equations.

### 3.2.3 Collision Histories

Collision histories are defined in the same way as section 2.2.3, which is a simplified version of the definition of collision trees in [38]. A collision history  $\Phi$  encodes the initial position and velocity of the tagged particle along with the list of collisions that it experiences.

**Definition 3.2.5.** The set of collision histories is defined by,

$$\mathcal{MT} := \{(x_0, v_0), (t_1, \nu_1, v_1), \dots, (t_n, \nu_n, v_n) : n \in \mathbb{N} \cup \{0\}, 0 < t_1 < \dots < t_n\}.$$

For a history  $\Phi \in \mathcal{MT}$ ,  $n$  denotes the number of collisions. The final collision plays an important role in this theory. We define  $\tau = \tau(\Phi)$ ,

$$\tau := \begin{cases} 0 & \text{if } n = 0, \\ t_n & \text{if } n \geq 1, \end{cases} \quad (3.2.10)$$

and for  $n \geq 1$  we use the notation  $(\tau, \nu, v') = (t_n, \nu_n, v_n)$ . Finally, for  $n \geq 1$ , we define  $\bar{\Phi}$  as the history  $\Phi$  but with the final collision removed. For example if  $\Phi = ((x_0, v_0), (\tau, \nu, v'))$  then  $\bar{\Phi} = ((x_0, v_0))$ .

For  $\Phi \in \mathcal{MT}$  define  $\Phi_0 = (x_0, v_0)$  and for  $1 \leq j \leq n$  define  $\Phi_j = (t_j, \nu_j, v_j)$ . We define a metric,  $d$ , on  $\mathcal{MT}$  as follows. For any  $\Phi, \Psi \in \mathcal{MT}$ ,

$$d(\Phi, \Psi) := \begin{cases} 1, & \text{if } n(\Phi) \neq n(\Psi) \\ \min \left\{ 1, \max_{0 \leq j \leq n} |\Phi_j - \Psi_j| \right\} & \text{else.} \end{cases}$$

For  $\Phi \in \mathcal{MT}$  and  $h > 0$  we define

$$B_h(\Phi) := \left\{ \Psi \in \mathcal{MT} : d(\Phi, \Psi) < \frac{h}{2} \right\}.$$

We note that for a given  $\varepsilon \geq 0$ , the realisation of  $\Phi$  at a time  $t \in [0, T]$  uniquely determines  $(x(t), v(t))$ , the position and velocity of the tagged particle, and  $(x_j(t), v_j(t))$ , the position and velocity of the  $j$  background particles involved in the history. We note that  $(x(t), v(t))$  is independent of  $\varepsilon$  (since regardless of  $\varepsilon$  the tagged particle has given velocities and collision times), but each  $(x_j(t), v_j(t))$  is  $\varepsilon$  dependent (since the relevant background particle must be  $\varepsilon$  from the tagged particle at the collision).

Further the realisation of  $\Phi$  gives information on the remaining  $N - n$  background particles, since we know that they have not interfered with the tagged particle.

### 3.2.4 The Propagation of Chaos

This tree history approach allows us to avoid the issue of proving the propagation of chaos explicitly. This approach was developed in [35] to circumvent the issues around the propagation of chaos by focusing on good histories or trees.

The idealised distribution  $P_t^\varepsilon$  considers that the particles are chaotic so the probability of seeing a background particle at  $(x, v)$  at time  $t$  is given exactly by  $g_0(x - tv, v)$ .

On the other hand for the empirical distribution no assumption of chaos is made and the particles evolve as described by the particle dynamics. Therefore the probability of seeing a background particle at  $(x, v)$  at time  $t$  is more involved than just  $g_0(x - tv, v)$  since we need to consider the effect of a background particle colliding, changing velocity and then arriving at  $(x, v)$  at time  $t$ .

This issue is resolved by considering only good collision histories. Good histories, defined precisely in definition 3.4.14, require, among other properties, that each background particle that the tagged particle collides with will not re-collide with the tagged particle up to time  $T$ . This means that if we restrict our attention to good histories then we know that there cannot be any re-collisions and so the distribution of the background particles is much clearer. For this reason we only investigate the properties of the empirical distribution  $\hat{P}_t^\varepsilon$  on this set of good histories.

It is then shown in proposition 3.5.5 that good histories have full measure, in the sense that the contribution of histories that are not good is vanishing as  $\varepsilon$  tends to zero.

Therefore to prove convergence between the idealised distribution and the empirical distribution, which is the key step to proving the main theorem, we only need to compare the idealised and empirical distributions on good histories and remark that the effect of histories that are not good is vanishing in the limit.

Hence the propagation of chaos is proved implicitly with this collision history method. The idealised distribution assumes chaos whereas the empirical distribution does not. By proving the convergence from the empirical distribution to the idealised distribution we prove the propagation of chaos implicitly.

We emphasise that good histories, due to their lack of re-collisions, mean that the propagation of chaos holds for the particles relevant for the tagged particle.



### 3.3 The Idealised Distribution

The idealised equation is the first of two Kolmogorov equations in this chapter. In this section we show that there exists a solution to the idealised equation and relate it to the solution of the linear Boltzmann equation. We construct a solution by first considering the probability of finding the tagged particle at a certain position and velocity such that it has not yet had any collisions. From this we iteratively define a function and check that it solves the idealised equation and that the required connection to the linear Boltzmann equation holds.

A significant problem in this section, and where we find the main difference to chapter 2, is showing we have the required evolution semigroup to solve the non-autonomous equation that describes the probability of finding the tagged particle such that it has not yet experience any collisions. In the autonomous case in the previous chapter we were able to quote specific semigroup results for the Boltzmann equation from [6]. However in this non-autonomous case we have to resort to more general evolution semigroup theory. This results in a number of technical results to check the various assumptions of the general theory.

In order to compare the solution of the idealised equation with the solution of the empirical equation, which is the main step in proving theorem 3.2.4, we consider an intermediate step by introducing a dependence on  $\varepsilon$  in the idealised equation. In order to be able to connect this  $\varepsilon$  dependent solution of the idealised equation to the linear Boltzmann equation we introduce an  $\varepsilon$  dependent linear Boltzmann equation. Similarly to (3.2.7) and (3.2.8), for  $\varepsilon \geq 0$ ,  $t \geq 0$  define  $Q_t^{\varepsilon,+}$  and  $Q_t^{\varepsilon,-} : L^1(U \times \mathbb{R}^3) \rightarrow L^1(U \times \mathbb{R}^3)$  by,

$$Q_t^{\varepsilon,+}[f](x, v) := \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} f(x, v') g_t(x + \varepsilon \nu, \bar{v}') [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu,$$

and,

$$Q_t^{\varepsilon,-}[f](x, v) := f(x, v) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x + \varepsilon \nu, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu.$$

Define  $Q_t^\varepsilon := Q_t^{\varepsilon,+} - Q_t^{\varepsilon,-}$ . Then the  $\varepsilon$  dependent non-autonomous linear Boltzmann equation is given by,

$$\begin{cases} \partial_t f_t^\varepsilon(x, v) &= -v \cdot \nabla_x f_t^\varepsilon(x, v) + Q_t^\varepsilon[f_t^\varepsilon](x, v), \\ f_{t=0}^\varepsilon(x, v) &= f_0(x, v). \end{cases} \quad (3.3.1)$$

For an intuitive description of the idealised equation and the terms it includes see the discussion in section 2.3. We can now state the idealised equation. For  $\varepsilon \geq 0$

consider,

$$\begin{cases} \partial_t P_t(\Phi) &= \mathcal{Q}_t^\varepsilon[P_t](\Phi) = \mathcal{Q}_t^{\varepsilon,+}[P_t](\Phi) - \mathcal{Q}_t^{\varepsilon,-}[P_t](\Phi), \\ P_0(\Phi) &= f_0(x_0, v_0) \mathbb{1}_{n(\Phi)=0}, \end{cases} \quad (3.3.2)$$

where,

$$\mathcal{Q}_t^{\varepsilon,+}[P_t](\Phi) := \begin{cases} \delta(t - \tau) P_\tau(\bar{\Phi}) g_\tau(x(\tau) + \varepsilon\nu, v') [(v(\tau^-) - v') \cdot \nu]_+ & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \end{cases} \quad (3.3.3)$$

$$\mathcal{Q}_t^{\varepsilon,-}[P_t](\Phi) := P_t(\Phi) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon\nu, \bar{v}) [(v(t) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu. \quad (3.3.4)$$

For a history  $\Phi \in \mathcal{MT}$ ,  $t \geq 0$  and  $\varepsilon \geq 0$  we introduce the notation,

$$L_t^\varepsilon(\Phi) := \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon\nu, \bar{v}) [(v(t) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu, \quad (3.3.5)$$

and note that this implies

$$\mathcal{Q}_t^{\varepsilon,-}[P_t](\Phi) = P_t(\Phi) L_t^\varepsilon(\Phi).$$

Moreover for any  $t \in [0, T]$  and for any  $\Omega \subset U \times \mathbb{R}^3$  define,

$$S_t(\Omega) := \{\Phi \in \mathcal{MT} : (x(t), v(t)) \in \Omega\}.$$

**Theorem 3.3.1.** *Suppose that  $f_0$  and  $g_0$  are tagged and background admissible respectively in the sense of definition 3.2.3. Then for all  $\varepsilon \geq 0$  there exists a solution  $P^\varepsilon : [0, T] \rightarrow L^1(\mathcal{MT})$  to (3.3.2) such that for all  $t \in [0, T]$ ,  $P_t^\varepsilon$  is a probability measure on  $\mathcal{MT}$ . Furthermore there exists a  $K > 0$ , independent of  $\varepsilon$  such that for any  $t \in [0, T]$ ,*

$$\int_{\mathcal{MT}} P_t^\varepsilon(\Phi) (1 + |v(\tau)|) d\Phi \leq K < \infty. \quad (3.3.6)$$

And for any  $\varepsilon \geq 0$ ,  $t \in [0, T]$  and any  $\Omega \subset U \times \mathbb{R}^3$  measurable,

$$\int_{\Omega} f_t^\varepsilon(x, v) dx dv = \int_{S_t(\Omega)} P_t^\varepsilon(\Phi) d\Phi, \quad (3.3.7)$$

where  $f_t^\varepsilon$  is a solution to (3.3.1). Finally, uniformly for  $t \in [0, T]$ ,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{MT}} |P_t^0(\Phi) - P_t^\varepsilon(\Phi)| d\Phi = 0. \quad (3.3.8)$$

From now we assume that  $f_0$  and  $g_0$  are tagged and background admissible respectively.

The rest of this section is devoted to proving theorem 3.3.1. We split this into a number of subsections. In the first subsection, 3.3.1, we prove that there exists a solution  $P_t^{\varepsilon,(0)}$  to the gainless linear Boltzmann equation and that this solution has a particular form given by an evolution semigroup  $U^\varepsilon$ . This subsection takes a number of technical lemmas in order to prove various semigroup properties. Then in subsection 3.3.2 we show that the  $\varepsilon$  dependent non-autonomous linear Boltzmann equation has a solution, in the evolution semigroup sense. Then in section 3.3.3 we construct  $P_t^\varepsilon$  and show that it indeed satisfies the properties of theorem 3.3.1. We finish this section by using theorem 3.3.1 to prove that the solution of the  $\varepsilon$  dependent non-autonomous linear Boltzmann equation is a probability measure.

### 3.3.1 The Evolution Semigroup

In this subsection we prove that there exists a solution to the  $\varepsilon$  dependent gainless linear Boltzmann equation (3.3.11) by following standard evolution semigroup theory as in [40]. This requires a number of technical results.

**Definition 3.3.2.** For any  $t \in [0, T]$  and any  $\varepsilon \geq 0$  define  $D(A^\varepsilon(t)), D(B^\varepsilon(t)) \subset L^1(U \times \mathbb{R}^3)$  by,

$$\begin{aligned} D(A^\varepsilon(t)) &:= \{f \in L^1(U \times \mathbb{R}^3) : v \cdot \nabla_x f(x, v) + Q_t^{\varepsilon,-}[f](x, v) \in L^1(U \times \mathbb{R}^3)\}, \\ D(B^\varepsilon(t)) &:= \{f \in L^1(U \times \mathbb{R}^3) : Q_t^{\varepsilon,+}[f](x, v) \in L^1(U \times \mathbb{R}^3)\}. \end{aligned}$$

Then define operators  $A^\varepsilon(t) : D(A^\varepsilon(t)) \rightarrow L^1(U \times \mathbb{R}^3)$  and  $B^\varepsilon(t) : D(B^\varepsilon(t)) \rightarrow L^1(U \times \mathbb{R}^3)$  by,

$$(A^\varepsilon(t)f)(x, v) := -v \cdot \nabla_x f(x, v) - Q_t^{\varepsilon,-}[f](x, v) \quad (3.3.9)$$

$$(B^\varepsilon(t)f)(x, v) := Q_t^{\varepsilon,+}[f](x, v). \quad (3.3.10)$$

**Proposition 3.3.3.** For  $\varepsilon \geq 0$  there exists a solution  $P^{\varepsilon,(0)} : [0, T] \rightarrow L^1(U \times \mathbb{R}^3)$  to the following equation,

$$\begin{cases} \partial_t P_t^{\varepsilon,(0)}(x, v) &= (A^\varepsilon(t)P_t^{\varepsilon,(0)})(x, v), \\ P_0^{\varepsilon,(0)}(x, v) &= f_0(x, v), \end{cases} \quad (3.3.11)$$

Moreover the solution is given by  $P_t^{\varepsilon,(0)} = U^\varepsilon(t, 0)f_0$ , where  $U^\varepsilon : [0, T] \times [0, T] \times L^1(U \times \mathbb{R}^3) \rightarrow L^1(U \times \mathbb{R}^3)$

$\mathbb{R}^3) \rightarrow L^1(U \times \mathbb{R}^3)$  is defined by,

$$(U^\varepsilon(t, s)f)(x, v) := \exp \left( - \int_s^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x + \varepsilon\nu - (t - \sigma)v, \bar{v})[(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) f(x - (t - s)v, v). \quad (3.3.12)$$

**Remark 3.3.4.**  $P_t^{\varepsilon, (0)}(x, v)$  can be thought of as the probability of finding the tagged particle at  $(x, v)$  such that it has not yet experienced any collisions.

To prove this proposition we aim to apply [40, Theorem 5.3.1], which gives that there exists a evolution semigroup defining the solution to (3.3.11). First we present lemmas checking that conditions (H1), (H2) and (H3) hold. This tells us that there exists a unique evolution semigroup satisfying (E1), (E2) and (E3). Next we show that  $U^\varepsilon(t, s)$  is a strongly continuous evolution semigroup and that it satisfies (E1), (E2) and (E3), so is indeed the evolution semigroup described by [40, Theorem 5.3.1]. This tells us that a solution to (3.3.11) is given by  $U^\varepsilon(t, 0)f_0$ .

**Lemma 3.3.5.** *For  $\varepsilon \geq 0$ ,  $A^\varepsilon$ , as defined in definition 3.3.2, satisfies condition (H1) of [40, Chapter 5].*

*Proof.* By [3, Theorem 10.4] we see that for  $t \geq 0$ ,  $A^\varepsilon(t)$  generates the  $C_0$  semigroup  $S_t^\varepsilon$  given by,

$$\begin{aligned} (S_t^\varepsilon(s)P)(x, v) &= \exp \left( - \int_0^s \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x + \varepsilon\nu - \sigma v, \bar{v})[(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) P(x - sv, v). \end{aligned} \quad (3.3.13)$$

Since each  $S_t^\varepsilon$  is a contraction semigroup we see that this is a stable family, which proves condition (H1) of Theorem 3.1 [40, Chapter 5].  $\square$

Define

$$Y := \{P \in L^1(U \times \mathbb{R}^3) : \text{for almost all } v \in \mathbb{R}^3, P(\cdot, v) \in W^{1,1}(U) \text{ and } \|P\|_Y < \infty\}, \quad (3.3.14)$$

where,

$$\|P\|_Y := \int_{U \times \mathbb{R}^3} (1 + |v|^2)|P(x, v)| + (1 + |v|)|\nabla_x P(x, v)| dx dv.$$

The following two lemmas, lemma 3.3.6 and lemma 3.3.7, are used to help prove that condition (H2) holds, which is shown in lemma 3.3.8.

**Lemma 3.3.6.** *For  $\varepsilon, t, s \geq 0$ ,  $Y$  is invariant under the map  $S_t^\varepsilon(s)$ .*

*Proof.* Let  $P \in Y$ . It is clear that for almost all  $v \in \mathbb{R}^3$ ,  $S_t^\varepsilon(s)P(\cdot, v) \in L^1(U)$ . Further, for each  $i = 1, 2, 3$ , and almost all  $x, v \in U \times \mathbb{R}^3$

$$\begin{aligned} & \partial_{x_i} \left( (S_t^\varepsilon(s)P)(x, v) \right) \\ &= - \left( \int_0^s \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \partial_{x_i} g_t(x + \varepsilon\nu - \sigma v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) (S_t^\varepsilon(s)P)(x, v) \\ & \quad + \exp \left( - \int_0^s \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x + \varepsilon\nu - \sigma v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) \partial_{x_i} P(x - sv, v). \end{aligned} \quad (3.3.15)$$

Since  $P \in Y$  and using (3.2.5) we can integrate each of these terms over  $U$ . Hence for almost all  $v \in \mathbb{R}^3$ ,  $(S_t^\varepsilon(s)P)(\cdot, v) \in W^{1,1}(U)$ . It remains to check that  $\|S_t^\varepsilon(s)P\|_Y < \infty$ . By bounding the exponential term in (3.3.13) by 1 we have,

$$\int_{U \times \mathbb{R}^3} (1 + |v|^2) |S_t^\varepsilon(s)P(x, v)| dx dv \leq \int_{U \times \mathbb{R}^3} (1 + |v|^2) |P(x, v)| dx dv \leq \|P\|_Y < \infty. \quad (3.3.16)$$

Further we note that by (3.2.5) for some  $C > 0$ ,

$$\begin{aligned} & \int_{U \times \mathbb{R}^3} (1 + |v|) \left( \int_0^s \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \partial_{x_i} g_t(x + \varepsilon\nu - \sigma v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) \\ & \quad (S_t^\varepsilon(s)P)(x, v) dx dv \\ & \leq \int_{U \times \mathbb{R}^3} (1 + |v|) \left( \int_0^s \int_{\mathbb{S}^2} (1 + |v|) M_1 d\nu d\sigma \right) P(x - sv, v) dx dv \\ & \leq C \int_{U \times \mathbb{R}^3} (1 + |v|^2) P(x - sv, v) dx dv \leq C \|P\|_Y < \infty. \end{aligned} \quad (3.3.17)$$

Also,

$$\begin{aligned} & \int_{U \times \mathbb{R}^3} (1 + |v|) \exp \left( - \int_0^s \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x + \varepsilon\nu - \sigma v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) \\ & \quad \partial_{x_i} P(x - sv, v) dx dv \\ & \leq \int_{U \times \mathbb{R}^3} (1 + |v|) \partial_{x_i} P(x - sv, v) dx dv \leq \|P\|_Y < \infty. \end{aligned} \quad (3.3.18)$$

Combining (3.3.15), (3.3.17) and (3.3.18) with (3.3.16) gives  $\|S_t^\varepsilon(s)P\|_Y < \infty$  as required. □

**Lemma 3.3.7.** *For  $\varepsilon, t \geq 0$ ,  $S_t^\varepsilon|_Y$ , the restriction of the semigroup  $S_t^\varepsilon$  to the space  $Y$ , is a  $C_0$  semigroup on  $Y$ .*

*Proof.* We know that  $S_t^\varepsilon$  is a semigroup in  $L^1(U \times \mathbb{R}^3)$  and  $Y$  is invariant under  $S_t^\varepsilon$  by lemma 3.3.6 so the only remaining property to check is that for any  $P \in Y$ ,

$$\lim_{s \downarrow 0} \|S_t^\varepsilon(s)P - P\|_Y = 0. \quad (3.3.19)$$

Let  $\eta \in C_c^\infty(U \times \mathbb{R}^3)$  be a test function and let  $\delta > 0$ . We show for  $s > 0$  sufficiently small,

$$\|S_t^\varepsilon(s)\eta - \eta\|_Y = \int_{U \times \mathbb{R}^3} (1 + |v|^2) |S_t^\varepsilon(s)\eta - \eta| + (1 + |v|) |\nabla_x(S_t^\varepsilon(s)\eta - \eta)| \, dx \, dv < \delta. \quad (3.3.20)$$

Since  $\eta \in C_c^\infty(U \times \mathbb{R}^3)$  there exists an  $R > 0$  such that for all  $|v| > R$ ,  $\eta(\cdot, v) = 0$ . By (3.2.3) we have for any  $|v| < R$ ,

$$\begin{aligned} & \int_0^s \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x + \varepsilon\nu - \sigma v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu \, d\sigma \\ & \leq \int_0^s \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \bar{g}(\bar{v}) [(v - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu \, d\sigma \\ & \leq \pi \int_0^s \int_{\mathbb{R}^3} \bar{g}(\bar{v}) (|v| + |\bar{v}|) \, d\bar{v} \, d\sigma \leq s\pi M_g(1 + R). \end{aligned}$$

Hence for  $|v| < R$ ,

$$\begin{aligned} & 1 - \exp \left( - \int_0^s \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x + \varepsilon\nu - \sigma v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu \, d\sigma \right) \\ & \leq 1 - \exp(-s\pi M_g(1 + R)), \end{aligned}$$

and this converges to zero as  $s$  converges to zero. Therefore,

$$\begin{aligned} & \int_{U \times \mathbb{R}^3} (1 + |v|^2) |S_t^\varepsilon(s)\eta - \eta| \, dx \, dv \\ & \leq \int_{U \times B_R(0)} (1 + |v|^2) \left( |\eta(x - sv, v) - \eta(x, v)| \right. \\ & \quad \left. + (1 - \exp(-s\pi M_g(1 + R))) |\eta(x, v)| \right) \, dx \, dv. \end{aligned} \quad (3.3.21)$$

Since  $\eta$  is continuous on  $U \times \mathbb{R}^3$ , is it uniformly continuous on  $U \times B_R(0)$  so we can make  $s$  sufficiently small so that this is less than  $\delta/3$ . Now by (3.2.5) we have for almost all  $x$ ,

$$\int_0^s \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} |\nabla_x g_t(x + \varepsilon\nu - \sigma v, \bar{v})| [(v - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu \, d\sigma \leq s\pi M_1(1 + |v|). \quad (3.3.22)$$

Hence for  $s$  sufficiently small,

$$\begin{aligned}
& \int_{U \times \mathbb{R}^3} (1 + |v|) \left| \int_0^s \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \nabla_x g_t(x + \varepsilon\nu - \sigma v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right| \\
& \quad |S_t^\varepsilon(s)\eta(x, v)| dx dv \\
& \leq s \int_{U \times B_R(0)} \pi M_1 (1 + |v|)^2 |\eta(x - sv, v)| dx dv < \delta/3.
\end{aligned} \tag{3.3.23}$$

Also, by a similar process to (3.3.21) we see that for  $s$  sufficiently small,

$$\int_{U \times \mathbb{R}^3} (1 + |v|) |S_t^\varepsilon(s) \nabla_x \eta(x, v) - \nabla_x \eta(x, v)| dx dv < \delta/3. \tag{3.3.24}$$

Together (3.3.23) and (3.3.24) give that for  $s$  sufficiently small,

$$\begin{aligned}
& \int_{U \times \mathbb{R}^3} (1 + |v|) |\nabla_x (S_t^\varepsilon(s)\eta - \eta)| dx dv \\
& = \int_{U \times \mathbb{R}^3} (1 + |v|) \left| \left( \int_0^s \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \nabla_x g_t(x + \varepsilon\nu - \sigma v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) \right. \\
& \quad \left. S_t^\varepsilon(s)\eta(x, v) + S_t^\varepsilon(s) \nabla_x \eta(x, v) - \nabla_x \eta(x, v) \right| dx dv \\
& < 2\delta/3.
\end{aligned}$$

Therefore with (3.3.21), we see that for  $s$  sufficiently small, (3.3.20) holds. Now let  $P \in Y$ . For  $\delta > 0$  there exists an  $\eta \in C_c^\infty(U \times \mathbb{R}^3)$  such that  $\|P - \eta\|_Y < \delta$ . Using this and (3.3.20) finally (3.3.19) can be proved.  $\square$

**Lemma 3.3.8.** *For  $\varepsilon \geq 0$ , condition (H2) of [40, Theorem 5.3.1] is satisfied for  $Y$  as defined above.*

*Proof.* Lemma 3.3.6 proves that  $Y$  is invariant under  $S_t^\varepsilon(s)$  and lemma 3.3.7 proves that  $S_t^\varepsilon|_Y$  is a  $C_0$  semigroup on  $Y$ . It remains to prove that  $A^\varepsilon|_Y$  is a stable family in  $Y$ . We use [40, Theorem 5.2.2]. By the calculations in the proof of lemma 3.3.6 we have that for any  $s, t, \varepsilon \geq 0$  there exists a  $C \geq 1$  such that for any  $P \in Y$ ,

$$\|S_t^\varepsilon(s)P\|_Y \leq C\|P\|_Y.$$

Since  $A^\varepsilon(t)$  is the generator of the  $C_0$  semigroup  $S_t^\varepsilon$ , [40, Theorem 1.5.2] gives that  $(0, \infty) \subset \rho(A^\varepsilon(t))$ . Hence to apply [40, Theorem 5.2.2] we need to show that there exists an  $M \geq 1$  and  $\omega \geq 0$  such that for any  $k \in \mathbb{N}$ , any sequence  $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq T$ ,

any list  $0 \leq s_1, \dots, s_k$  and any  $P \in Y$ ,

$$\left\| \prod_{j=1}^k S_{t_j}^\varepsilon(s_j) P \right\|_Y \leq M \exp \left( \omega \sum_{j=1}^k s_j \right) \|P\|_Y. \quad (3.3.25)$$

To that aim fix  $k \in \mathbb{N}$ ,  $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq T$  and  $0 \leq s_1, \dots, s_k$  and  $P \in Y$ . Define  $\Pi P := \prod_{j=1}^k S_{t_j}^\varepsilon(s_j) P$ . By repeatedly applying (3.3.13) we see that,

$$\begin{aligned} \Pi P(x, v) = & \exp \left( - \int_0^{s_1} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_{t_1}(x + \varepsilon \nu - \sigma v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right. \\ & - \int_{s_1}^{s_1+s_2} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_{t_2}(x + \varepsilon \nu - \sigma v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \\ & - \dots \\ & \left. - \int_{s_1+\dots+s_{k-1}}^{s_1+\dots+s_k} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_{t_k}(x + \varepsilon \nu - \sigma v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) \\ & P(x - \sum_{j=1}^k s_j v, v). \end{aligned}$$

Denoting the expression inside the exponential by  $-W$  we have, by the same calculation as in (3.3.22), for almost all  $x \in U$ ,

$$\begin{aligned} |\nabla_x W| \leq & \int_0^{s_1} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} |\nabla_x g_{t_1}(x + \varepsilon \nu - \sigma v, \bar{v})| [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \\ & + \int_{s_1}^{s_1+s_2} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} |\nabla_x g_{t_2}(x + \varepsilon \nu - \sigma v, \bar{v})| [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \\ & + \dots \\ & + \int_{s_1+\dots+s_{k-1}}^{s_1+\dots+s_k} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} |\nabla_x g_{t_k}(x + \varepsilon \nu - \sigma v, \bar{v})| [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \\ \leq & \pi M_1 (1 + |v|) \sum_{j=1}^k s_j. \end{aligned}$$



Hence, by bounding  $\exp(-W) \leq 1$  and using that for any  $s \geq 0$ ,  $s \leq \exp(s)$  we have,

$$\begin{aligned}
\|\Pi P\|_Y &= \int_{U \times \mathbb{R}^3} (1 + |v|^2) |\Pi P(x, v)| + (1 + |v|) |\nabla_x (\Pi P)(x, v)| \, dx \, dv \\
&\leq \|P\|_Y + \int_{U \times \mathbb{R}^3} (1 + |v|) (|\nabla_x W| |\Pi P(x, v)| + |\Pi \nabla_x P(x, v)|) \, dx \, dv \\
&\leq \|P\|_Y + \int_{U \times \mathbb{R}^3} (1 + |v|) \left( \pi M_1 (1 + |v|) \sum_{j=1}^k s_j |P(x, v)| \right) \, dx \, dv + \|P\|_Y \\
&\leq 2\|P\|_Y + 2\pi M_1 \sum_{j=1}^k s_j \int_{U \times \mathbb{R}^3} (1 + |v|^2) |P(x, v)| \, dx \, dv \\
&\leq 2\pi(M_1 + 1) \|P\|_Y + 2\pi(M_1 + 1) \exp \left( \sum_{j=1}^k s_j \right) \|P\|_Y \\
&\leq 4\pi(M_1 + 1) \exp \left( \sum_{j=1}^k s_j \right) \|P\|_Y.
\end{aligned}$$

Hence we see that for  $M = 4\pi(M_1 + 1)$  and  $\omega = 1$  (3.3.25) holds. Thus we can apply [40, Theorem 5.2.2] which proves that  $A^\varepsilon(t)|_Y$  is a stable family in  $Y$ , which completes the proof of the lemma.  $\square$

Having proved condition (H2) in the previous lemma we now move on to proving that condition (H3) holds. Firstly we prove a technical bounding lemma.

**Lemma 3.3.9.** *There exists a  $C > 0$  such that for any  $\varepsilon \geq 0$ ,  $R \geq 1$ ,  $t, s \geq 0$ ,  $v \in \mathbb{R}^3$  and almost all  $x, y \in U$ ,*

$$\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} |g_t(x + \varepsilon \nu, \bar{v}) - g_s(y + \varepsilon \nu, \bar{v})| [(v - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu \leq C(1 + |v|) \left( \frac{1}{R} + R^5 (|x - y|^\alpha + |t - s|^\alpha) \right).$$

*Proof.* Let  $\varepsilon \geq 0$ . Firstly by (3.2.3),

$$\begin{aligned}
\int_{\mathbb{R}^3 \setminus B_R(0)} \bar{g}(\bar{v}) (1 + |\bar{v}|) \, d\bar{v} &\leq \int_{\mathbb{R}^3 \setminus B_R(0)} \bar{g}(\bar{v}) \left( \frac{|\bar{v}|}{R} + \frac{|\bar{v}|^2}{R} \right) \, d\bar{v} \\
&\leq \frac{1}{R} \int_{\mathbb{R}^3} \bar{g}(\bar{v}) (|\bar{v}| + |\bar{v}|^2) \, d\bar{v} \leq \frac{2M_g}{R},
\end{aligned}$$

where  $B_R(0)$  denotes the ball of radius  $R$  around 0 in  $\mathbb{R}^3$ . Hence

$$\begin{aligned}
& \int_{\mathbb{S}^2} \int_{\mathbb{R}^3 \setminus B_R(0)} |g_t(x + \varepsilon\nu, \bar{v}) - g_s(y + \varepsilon\nu, \bar{v})| [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \\
& \leq \int_{\mathbb{S}^2} \int_{\mathbb{R}^3 \setminus B_R(0)} (g_t(x + \varepsilon\nu, \bar{v}) + g_s(y + \varepsilon\nu, \bar{v})) (|v| + |\bar{v}|) d\bar{v} d\nu \\
& \leq \int_{\mathbb{S}^2} \int_{\mathbb{R}^3 \setminus B_R(0)} 2\bar{g}(\bar{v}) (|v| + |\bar{v}|) d\bar{v} d\nu \leq 4\pi \frac{M_g}{R} (1 + |v|). \tag{3.3.26}
\end{aligned}$$

Further by (3.2.6), for any  $\nu \in \mathbb{S}^2$ , for almost all  $\bar{v} \in B_R(0)$  and almost all  $x, y \in U$  we have, since  $0 < \alpha \leq 1$ ,

$$\begin{aligned}
|g_t(x + \varepsilon\nu, \bar{v}) - g_s(y + \varepsilon\nu, \bar{v})| & \leq M|x - y - (t - s)\bar{v}|^\alpha \leq M(|x - y|^\alpha + |(t - s)\bar{v}|^\alpha) \\
& \leq M(|x - y|^\alpha + R^\alpha |t - s|^\alpha) \leq MR(|x - y|^\alpha + |t - s|^\alpha).
\end{aligned}$$

Hence,

$$\begin{aligned}
& \int_{\mathbb{S}^2} \int_{B_R(0)} |g_t(x + \varepsilon\nu, \bar{v}) - g_s(y + \varepsilon\nu, \bar{v})| [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \\
& \leq \int_{\mathbb{S}^2} \int_{B_R(0)} MR(|x - y|^\alpha + |t - s|^\alpha) (|v| + |\bar{v}|) d\bar{v} d\nu \\
& \leq \int_{\mathbb{S}^2} \int_{B_R(0)} MR(|x - y|^\alpha + |t - s|^\alpha) (|v| + R) d\bar{v} d\nu \\
& \leq \frac{4}{3}\pi R^3 \times 2\pi \times MR(|x - y|^\alpha + |t - s|^\alpha) \times R(1 + |v|) \\
& = \frac{8}{3}\pi^2 MR^5 (1 + |v|) (|x - y|^\alpha + |t - s|^\alpha). \tag{3.3.27}
\end{aligned}$$

Together (3.3.26) and (3.3.27) give,

$$\begin{aligned}
& \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} |g_t(x + \varepsilon\nu, \bar{v}) - g_s(y + \varepsilon\nu, \bar{v})| [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \\
& = \int_{\mathbb{S}^2} \int_{B_R(0)} |g_t(x + \varepsilon\nu, \bar{v}) - g_s(y + \varepsilon\nu, \bar{v})| [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \\
& \quad + \int_{\mathbb{S}^2} \int_{\mathbb{R}^3 \setminus B_R(0)} |g_t(x + \varepsilon\nu, \bar{v}) - g_s(y + \varepsilon\nu, \bar{v})| [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \\
& \leq 4\pi \frac{M_g}{R} (1 + |v|) + \frac{8}{3}\pi^2 R^5 M (|x - y|^\alpha + |t - s|^\alpha) (1 + |v|) \\
& \leq C(1 + |v|) \left( \frac{1}{R} + R^5 (|x - y|^\alpha + |t - s|^\alpha) \right),
\end{aligned}$$

where,

$$C := \max \left\{ 4\pi M_g, \frac{8}{3}\pi^2 M \right\}.$$

□

**Lemma 3.3.10.** *For  $\varepsilon \geq 0$ , condition (H3) of [40, Theorem 5.3.1] is satisfied for  $Y$  as defined above.*

*Proof.* Let  $P \in Y$ . Notice that, by (3.2.2) and (3.2.3),

$$\begin{aligned} \|A^\varepsilon(t)P\| &= \int_{U \times \mathbb{R}^3} \left| -\nu \cdot \nabla_x P(x, v) \right. \\ &\quad \left. - P(x, v) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x + \varepsilon\nu, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \right| dx dv \\ &\leq \int_{U \times \mathbb{R}^3} (1 + |v|) |\nabla_x P(x, v)| + |P(x, v)| \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \bar{g}(\bar{v}) (|v| + |\bar{v}|) d\bar{v} d\nu dx dv \\ &\leq \int_{U \times \mathbb{R}^3} (1 + |v|) |\nabla_x P(x, v)| + 2\pi M_g (1 + |v|) |P(x, v)| dx dv \\ &\leq C \|P\|_Y, \end{aligned}$$

for some  $C > 0$ . Hence  $Y \subset D(A^\varepsilon(t))$  and  $A^\varepsilon(t)$  is bounded as a map  $Y \rightarrow X$ . It remains to prove that  $t \mapsto A^\varepsilon(t)$  is continuous in the  $B(Y, X)$  norm. Let  $P \in Y$ ,  $t \geq 0$  and  $\delta > 0$ . We seek an  $\eta > 0$  such that for all  $s \geq 0$  with  $|t - s| < \eta$ , we have

$$\|A^\varepsilon(t)P - A^\varepsilon(s)P\|_X = \int_{U \times \mathbb{R}^3} |A^\varepsilon(t)P - A^\varepsilon(s)P| dx dv \leq \delta.$$

Now by the definition of  $A^\varepsilon$ , in definition 3.3.2,

$$\begin{aligned} &\int_{U \times \mathbb{R}^3} |A^\varepsilon(t)P - A^\varepsilon(s)P| dx dv \\ &\leq \int_{U \times \mathbb{R}^3} P(x, v) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} |g_t(x + \varepsilon\nu, \bar{v}) - g_s(x + \varepsilon\nu, \bar{v})| [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu dx dv. \end{aligned} \tag{3.3.28}$$

Now take  $R \geq 1$  sufficiently large such that  $R > 2C/\delta$ , where  $C$  is as in lemma 3.3.9. Further take  $\eta > 0$  sufficiently small so that,  $CR^5\eta^\alpha < \delta/2$ . Then lemma 3.3.9 gives that for any  $s$  such that  $|t - s|^\alpha < \eta$  and for almost all  $x \in U$ ,

$$\begin{aligned} &\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} |g_t(x + \varepsilon\nu, \bar{v}) - g_s(x + \varepsilon\nu, \bar{v})| [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \\ &\leq C(1 + |v|) \left( \frac{1}{R} + R^5 |t - s|^\alpha \right) \leq (1 + |v|) \delta. \end{aligned}$$

Hence substituting this into (3.3.28),

$$\int_{U \times \mathbb{R}^3} |A^\varepsilon(t)P - A^\varepsilon(s)P| \, dx \, dv \leq \delta \int_{U \times \mathbb{R}^3} (1 + |v|)P(x, v) \, dx \, dv.$$

Taking the supremum over all  $\|P\|_Y \leq 1$  gives that  $\|A^\varepsilon(t) - A^\varepsilon(s)\|_{B(Y, X)} \leq \delta$  as required.  $\square$

The above lemmas have proved that conditions (H1), (H2) and (H3) hold. We now prove that the evolution semigroup that results from [40, Theorem 5.3.1] is indeed  $U^\varepsilon$  as defined in (3.3.12). We first show in the following lemma that  $U^\varepsilon$  is indeed an evolution semigroup.

**Lemma 3.3.11.** *Let  $U^\varepsilon$  be as in (3.3.12).  $U^\varepsilon$  is an exponentially bounded evolution family on  $L^1(U \times \mathbb{R}^3)$ .*

*Proof.* We use [39, definition 3.1]. It is clear to see that  $U^\varepsilon(s, s)$  is the identity operator. Further, for  $0 \leq s \leq r \leq t$  and  $f \in L^1(U \times \mathbb{R}^3)$  we have by (3.3.12),

$$\begin{aligned} U^\varepsilon(t, r)U^\varepsilon(r, s)f(x, v) &= U^\varepsilon(t, r) \exp \left( - \int_s^r \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x + \varepsilon\nu - (r - \sigma)v, \bar{v})[(v - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu \, d\sigma \right) \\ &\quad f(x - (r - s)v, v) \\ &= \exp \left( - \int_s^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x + \varepsilon\nu - (t - \sigma)v, \bar{v})[(v - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu \, d\sigma \right) \\ &\quad f(x - (t - s)v, v) \\ &= U^\varepsilon(t, s)f(x, v). \end{aligned}$$

As for exponential boundedness it easily follows with  $M = 1, \omega = 0$  by bounding the exponential term in  $U^\varepsilon$  by 1.  $\square$

In the following proposition we now prove that  $U^\varepsilon$  is indeed strongly continuous.

**Proposition 3.3.12.** *The evolution family  $U^\varepsilon$  is strongly continuous.*

To prove this proposition we use part 2 of [39, Proposition 3.2]. In the following lemmas we prove that iii) holds, that is, uniformly for  $0 \leq s \leq t$  in compact subsets,

- a)  $\lim_{s \uparrow t} U^\varepsilon(t, s)f = f$  for all  $f \in L^1(U \times \mathbb{R}^3)$
- b) for each  $s, f$  the mapping  $[s, \infty) \ni t \rightarrow U^\varepsilon(t, s)f$  is continuous and,
- c)  $\|U^\varepsilon(t, s)\|$  is bounded.

The proposition gives that this is equivalent to i), strong continuity. We note that c) has been proved in lemma 3.3.11. We prove a) and b) separately in the following two lemmas.

**Lemma 3.3.13.** *For all  $f \in L^1(U \times \mathbb{R}^3)$ ,*

$$\lim_{s \uparrow t} U^\varepsilon(t, s) f = f,$$

*uniformly for  $0 \leq s \leq t \leq T$ .*

To simplify notation here define:

$$E^\varepsilon(t, s, x, v) := \exp \left( - \int_s^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x + \varepsilon \nu - (t - \sigma)v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right). \quad (3.3.29)$$

*Proof.* Let  $\eta \in C_c^\infty(U \times \mathbb{R}^3)$ ,  $t \geq 0$  and  $\delta > 0$ . We show that for  $0 \leq s \leq t$  sufficiently close to  $t$ ,

$$\int_{U \times \mathbb{R}^3} |U^\varepsilon(t, s) \eta(x, v) - \eta(x, v)| dx dv < \delta. \quad (3.3.30)$$

Since  $\eta \in C_c^\infty(U \times \mathbb{R}^3)$  there exists an  $R > 0$  such that for all  $|v| > R$ ,  $\eta(\cdot, v) = 0$ . By (3.2.3) we have, for any  $v \in \mathbb{R}$  with  $|v| < R$ ,

$$\begin{aligned} & \int_s^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x + \varepsilon \nu - (t - \sigma)v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \\ & \leq \int_s^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \bar{g}(\bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \\ & \leq (t - s) \pi M_g (1 + R). \end{aligned} \quad (3.3.31)$$

This implies for  $t - s$  sufficiently small,

$$\begin{aligned} & \int_{U \times \mathbb{R}^3} |\eta(x - (t - s)v, v)| (1 - E^\varepsilon(t, s, x, v)) dx dv \\ & \leq \left( 1 - \exp(-(t - s) \pi M_g (1 + R)) \right) \int_{U \times B_R(0)} |\eta(x - (t - s)v, v)| dx dv < \delta/2. \end{aligned} \quad (3.3.32)$$

By the uniform continuity of  $\eta$  on  $U \times B_R(0)$ ,

$$\begin{aligned} & \int_{U \times \mathbb{R}^3} |\eta(x - (t - s)v, v) - \eta(x, v)| dx dv = \int_{U \times B_R(0)} |\eta(x - (t - s)v, v) - \eta(x, v)| dx dv \\ & < \delta/2. \end{aligned} \quad (3.3.33)$$

Hence by (3.3.32) and (3.3.33) for  $t - s$  sufficiently small,

$$\begin{aligned}
& \int_{U \times \mathbb{R}^3} |U^\varepsilon(t, s)\eta(x, v) - \eta(x, v)| \, dx \, dv \\
& \leq \int_{U \times \mathbb{R}^3} |E^\varepsilon(t, s, x, v)\eta(x - (t - s)v, v) - \eta(x - (t - s)v, v)| \\
& \quad + |\eta(x - (t - s)v, v) - \eta(x, v)| \, dx \, dv \\
& < \delta.
\end{aligned}$$

This proves (3.3.30). Now for a general  $f \in L^1(U \times \mathbb{R}^3)$  there exists an  $\eta \in C_c^\infty(U \times \mathbb{R}^3)$  such that,

$$\int_{U \times \mathbb{R}^3} |f(x, v) - \eta(x, v)| \, dx \, dv < \delta. \quad (3.3.34)$$

The required result follows by (3.3.34) and comparing  $f$  and  $U^\varepsilon(t, s)f$  with  $\eta$  and  $U^\varepsilon(t, s)\eta$  respectively.  $\square$

**Lemma 3.3.14.** *For any  $0 \leq s \leq t$ ,  $f \in L^1(U \times \mathbb{R}^3)$ , the mapping  $[s, \infty) \ni t \rightarrow U^\varepsilon(t, s)f$  is continuous.*

*Proof.* Fix  $f \in L^1(U \times \mathbb{R}^3)$  and  $\delta > 0$ . Let  $h > 0$ . By lemma 3.3.11  $U^\varepsilon$  is an evolution family so,

$$U^\varepsilon(t + h, s)f - U^\varepsilon(t, s)f = U^\varepsilon(t + h, t)U^\varepsilon(t, s)f - U^\varepsilon(t, s)f = U^\varepsilon(t + h, t)g - g,$$

where  $g = U^\varepsilon(t, s)f$ . Since  $g \in L^1(U \times \mathbb{R}^3)$  we can follow the proof of lemma 3.3.13 to prove that for  $h$  sufficiently small,

$$\int_{U \times \mathbb{R}^3} |U^\varepsilon(t + h, t)g(x, v) - g(x, v)| \, dx \, dv < \delta.$$

It remains to prove that

$$\lim_{h \downarrow 0} U^\varepsilon(t - h, s)f = U^\varepsilon(t, s)f.$$

Fix  $\delta > 0$ . Let  $h > 0$ . Then using (3.3.29),

$$\begin{aligned}
& \int_{U \times \mathbb{R}^3} |U^\varepsilon(t, s)f(x, v) - U^\varepsilon(t - h, s)f(x, v)| \, dx \, dv \\
&= \int_{U \times \mathbb{R}^3} |E^\varepsilon(t, s, x, v)f(x - (t - s)v, v) \\
&\quad - E^\varepsilon(t - h, s, x, v)f(x - (t - h - s)v, v)| \, dx \, dv \\
&\leq \int_{U \times \mathbb{R}^3} E^\varepsilon(t, s, x, v)|f(x - (t - s)v, v) - f(x - (t - h - s)v, v)| \\
&\quad + |E^\varepsilon(t - h, s, x, v) - E^\varepsilon(t, s, x, v)||f(x - (t - h - s)v, v)| \, dx \, dv \\
&= I_1 + I_2. \tag{3.3.35}
\end{aligned}$$

Now since  $E^\varepsilon(t, s, x, v) \leq 1$  we have that,

$$\begin{aligned}
I_1 &= \int_{U \times \mathbb{R}^3} E^\varepsilon(t, s, x, v)|f(x - (t - s)v, v) - f(x - (t - h - s)v, v)| \, dx \, dv \\
&\leq \int_{U \times \mathbb{R}^3} |f(x - (t - s)v, v) - f(x - (t - h - s)v, v)| \, dx \, dv.
\end{aligned}$$

We can make this less than  $\delta/2$  by approximating  $f$  with a test function  $\eta \in C_c^\infty(U \times \mathbb{R}^3)$  as in the above lemma. We now look to  $I_2$ . Firstly since  $f \in L^1(U \times \mathbb{R}^3)$  there exists an  $R > 0$  such that,

$$\int_{U \times \mathbb{R}^3 \setminus B_R(0)} f(x, v) \, dx \, dv < \frac{\delta}{8}.$$

Hence,

$$\begin{aligned}
I_2 &= \int_{U \times B_R(0)} |E^\varepsilon(t - h, s, x, v) - E^\varepsilon(t, s, x, v)||f(x - (t - h - s)v, v)| \, dx \, dv \\
&\quad + \int_{U \times \mathbb{R}^3 \setminus B_R(0)} |E^\varepsilon(t - h, s, x, v) - E^\varepsilon(t, s, x, v)||f(x - (t - h - s)v, v)| \, dx \, dv \\
&< \int_{U \times B_R(0)} |E^\varepsilon(t - h, s, x, v) - E^\varepsilon(t, s, x, v)||f(x - (t - h - s)v, v)| \, dx \, dv + \frac{\delta}{4}. \tag{3.3.36}
\end{aligned}$$

By the mean value theorem for any  $\alpha, \beta \leq 0$  there exists an  $\theta \in (\alpha, \beta) \cup (\beta, \alpha)$  such that,

$$\left| \frac{\exp(\alpha) - \exp(\beta)}{\alpha - \beta} \right| = \exp(\theta),$$

hence,

$$|\exp(\alpha) - \exp(\beta)| \leq |\alpha - \beta|. \tag{3.3.37}$$

By lemma 3.3.9, for any  $R_2 \geq 1$  and for almost all  $x \in U$ ,

$$\begin{aligned}
& \left| \exp \left( - \int_s^{t-h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x + \varepsilon\nu - (t-h-\sigma)v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) \right. \\
& \quad \left. - \exp \left( - \int_s^{t-h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x + \varepsilon\nu - (t-\sigma)v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) \right| \\
& \leq \int_s^{t-h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} |g_\sigma(x + \varepsilon\nu - (t-h-\sigma)v, \bar{v}) \\
& \quad - g_\sigma(x + \varepsilon\nu - (t-\sigma)v, \bar{v})| [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \\
& \leq \int_s^{t-h} C(1 + |v|) \left( \frac{1}{R_2} + R_2^5 h^\alpha \right) d\sigma \leq (t-h-s) C(1 + |v|) \left( \frac{1}{R_2} + R_2^5 h^\alpha \right) \\
& \leq TC(1 + |v|) \left( \frac{1}{R_2} + R_2^5 h^\alpha \right).
\end{aligned}$$

Hence, by a similar calculation to (3.3.31),

$$\begin{aligned}
& |E^\varepsilon(t-h, s, x, v) - E^\varepsilon(t, s, x, v)| \\
& = \left| \exp \left( - \int_s^{t-h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x + \varepsilon\nu - (t-h-\sigma)v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) \right. \\
& \quad \left. - \exp \left( - \int_s^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x + \varepsilon\nu - (t-\sigma)v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) \right| \\
& \leq \left| \exp \left( - \int_s^{t-h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x + \varepsilon\nu - (t-h-\sigma)v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) \right. \\
& \quad \left. - \exp \left( - \int_s^{t-h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x + \varepsilon\nu - (t-\sigma)v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) \right| \\
& \quad + \exp \left( - \int_s^{t-h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x + \varepsilon\nu - (t-\sigma)v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) \\
& \quad \left( 1 - \exp \left( - \int_{t-h}^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x + \varepsilon\nu - (t-\sigma)v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) \right) \\
& \leq TC(1 + |v|) \left( \frac{1}{R_2} + R_2^5 h^\alpha \right) \\
& \quad + \left( 1 - \exp \left( - \int_{t-h}^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x + \varepsilon\nu - (t-\sigma)v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu d\sigma \right) \right) \\
& \leq TC(1 + |v|) \left( \frac{1}{R_2} + R_2^5 h^\alpha \right) + 1 - \exp(-h\pi M_g(1 + |v|)).
\end{aligned}$$



This gives that,

$$\begin{aligned}
& \int_{U \times B_R(0)} |E^\varepsilon(t-h, s, x, v) - E^\varepsilon(t, s, x, v)| |f(x - (t-h-s)v, v)| \, dx \, dv \\
& \leq \int_{U \times B_R(0)} \left( TC(1+|v|) \left( \frac{1}{R_2} + R_2^5 h^\alpha \right) + 1 - \exp(-h\pi M_g(1+|v|)) \right) \\
& \quad |f(x - (t-h-s)v, v)| \, dx \, dv \\
& \leq \left( TC(1+R) \left( \frac{1}{R_2} + R_2^5 h^\alpha \right) + 1 - \exp(-h\pi M_g(1+R)) \right) \\
& \quad \int_{U \times B_R(0)} |f(x - (t-h-s)v, v)| \, dx \, dv \\
& \leq \left( TC(1+R) \left( \frac{1}{R_2} + R_2^5 h^\alpha \right) + 1 - \exp(-h\pi M_g(1+R)) \right) \|f\|.
\end{aligned}$$

Now take  $R_2 \geq 1$  sufficiently large such that,

$$\frac{TC(1+R)\|f\|}{R_2} < \frac{\delta}{12},$$

and  $h > 0$  sufficiently small so that both,

$$TC(1+R)R_2^5\|f\|h^\alpha < \frac{\delta}{12} \text{ and, } 1 - \exp(-h\pi M_g(1+R))\|f\| < \frac{\delta}{12}.$$

Hence,

$$\int_{U \times B_R(0)} |E^\varepsilon(t-h, s, x, v) - E^\varepsilon(t, s, x, v)| |f(x - (t-h-s)v, v)| \, dx \, dv < \frac{\delta}{4}.$$

Substituting this into (3.3.36) gives  $I_2 < \delta/2$ . Returning to (3.3.35) this gives for  $h > 0$  sufficiently small,

$$\int_{U \times \mathbb{R}^3} |U^\varepsilon(t, s)f(x, v) - U^\varepsilon(t-h, s)f(x, v)| \, dx \, dv < \delta,$$

which completes the proof of the lemma.  $\square$

*Proof of proposition 3.3.12.* This proposition follows from lemma 3.3.13 and lemma 3.3.14.  $\square$

Finally to prove proposition 3.3.3 it remains to prove that  $U^\varepsilon$  satisfies the properties (E1), (E2) and (E3).

**Proposition 3.3.15.** *The evolution semigroup  $U^\varepsilon$  satisfies the properties (E1), (E2) and (E3) of [40, Theorem 5.3.1].*

*Proof.* By bounding the exponential term by 1 it is clear that (E1) holds with  $M = 1, \omega = 0$ . Now let  $P \in Y$  and  $\Omega \subset U \times \mathbb{R}^3$  be measurable. Then,

$$\int_{\Omega} \partial_t|_{t=s} P(x - (t-s)v, v) \, dx \, dv = \int_{\Omega} -v \cdot \nabla_x P(x, v) \, dx \, dv.$$

And using (3.3.29) we have,

$$\begin{aligned} & \int_{\Omega} P(x, v) \partial_t|_{t=s} E^\varepsilon(t, s, x, v) \, dx \, dv \\ &= \int_{\Omega} P(x, v) \left( - \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x + \varepsilon\nu, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu \Big|_{t=s} \right. \\ & \quad \left. - v \cdot \int_s^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \nabla_x g_\sigma(x + \varepsilon\nu - (t - \sigma)v, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu \, d\sigma \Big|_{t=s} \right) \, dx \, dv \\ &= - \int_{\Omega} P(x, v) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_s(x + \varepsilon\nu, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu \, dx \, dv. \end{aligned}$$

Hence,

$$\begin{aligned} & \int_{\Omega} \partial_t|_{t=s} (U^\varepsilon(t, s)P)(x, v) \, dx \, dv \\ &= \int_{\Omega} P(x, v) \partial_t|_{t=s} E^\varepsilon(t, s, x, v) + E^\varepsilon(s, s, x, v) \partial_t|_{t=s} P(x - (t-s)v, v) \, dx \, dv \\ &= \int_{\Omega} -v \cdot \nabla_x P(x, v) - P(x, v) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_s(x + \varepsilon\nu, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu \, dx \, dv \\ &= \int_{\Omega} A^\varepsilon(s) P(x, v) \, dx \, dv. \end{aligned}$$

This proves (E2). Further,

$$\begin{aligned} & \int_{\Omega} E^\varepsilon(t, s, x, v) \partial_s P(x - (t-s)v, v) \, dx \, dv \\ &= \int_{\Omega} E^\varepsilon(t, s, x, v) v \cdot \nabla_x P(x - (t-s)v, v) \, dx \, dv. \end{aligned}$$

And,

$$\begin{aligned} & \int_{\Omega} P(x - (t-s)v, v) \partial_s E^\varepsilon(t, s, x, v) \, dx \, dv \\ &= \int_{\Omega} P(x - (t-s)v, v) E^\varepsilon(t, s, x, v) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_s(x + \varepsilon\nu - (t-s)v, \bar{v}) \\ & \quad [(v - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu \, dx \, dv. \end{aligned}$$

Hence,

$$\begin{aligned}
& \int_{\Omega} \partial_s (U^\varepsilon(t, s)P)(x, v) \, dx \, dv \\
&= \int_{\Omega} E^\varepsilon(t, s, x, v) \partial_s P(x - (t - s)v, v) + P(x - (t - s)v, v) \partial_s E^\varepsilon(t, s, x, v) \, dx \, dv \\
&= \int_{\Omega} E^\varepsilon(t, s, x, v) v \cdot \nabla_x P(x - (t - s)v, v) \\
&\quad + E^\varepsilon(t, s, x, v) P(x - (t - s)v, v) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_s(x + \varepsilon\nu - (t - s)v, \bar{v}) \\
&\quad \quad [(v - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu \, dx \, dv \\
&= \int_{\Omega} -U^\varepsilon(t, s) A^\varepsilon(s) P(x, v) \, dx \, dv.
\end{aligned}$$

This proves (E3) which completes the proof of the lemma.  $\square$

We can finally now combine all the results in this subsection to prove proposition 3.3.3.

*Proof of proposition 3.3.3.* Let  $\varepsilon \geq 0$ . By lemmas 3.3.5, 3.3.8 and 3.3.10 we can apply [40, Theorem 5.3.1]. This gives that there exists a unique evolution semigroup satisfying (E1), (E2), (E3). By lemma 3.3.11,  $U^\varepsilon$  is an exponentially bounded evolution family and by proposition 3.3.12 it is strongly continuous. By proposition 3.3.15,  $U^\varepsilon$  satisfies these conditions and hence the solution is given by  $P_t^{\varepsilon, (0)} = U^\varepsilon(t, 0)f_0$  as required.  $\square$

### 3.3.2 Existence of Non-Autonomous Linear Boltzmann Solution

In this subsection we prove that there exists a solution to the  $\varepsilon$  dependent non-autonomous linear Boltzmann equation (3.3.1). We prove the result by adapting the method of [4].

**Proposition 3.3.16.** *For  $\varepsilon \geq 0$  there exists a solution  $f^\varepsilon : [0, T] \rightarrow L^1(U \times \mathbb{R}^3)$  to the non-autonomous linear Boltzmann equation (3.2.9). Moreover there exists a  $K > 0$  such that for any  $t \in [0, T]$  and any  $\varepsilon \geq 0$ ,*

$$\int_{U \times \mathbb{R}^3} f_t^\varepsilon(x, v) (1 + |v|) \, dx \, dv \leq K. \tag{3.3.38}$$

**Remark 3.3.17.** Later, in proposition 3.3.37, we are able to show that for any  $\varepsilon \geq 0$  and any  $t \in [0, T]$ ,  $f_t^\varepsilon$  is a probability measure on  $U \times \mathbb{R}^3$  and that  $f_t^\varepsilon$  converges in total variation to  $f_t^0$  uniformly for  $t \in [0, T]$ .

We first introduce some notation. For  $\varepsilon, t \geq 0$ ,  $(x, v) \in U \times \mathbb{R}^3$  and  $v_* \neq v \in \mathbb{R}^3$  define,

$$\begin{aligned}\Sigma_t^\varepsilon(x, v) &:= \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x + \varepsilon\nu, \bar{v})[(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu, \text{ and} \\ k_t^\varepsilon(x, v, v_*) &:= \frac{1}{|v - v_*|} \int_{E_{vv_*}} g_t(x + \varepsilon\nu, w) dw,\end{aligned}$$

where  $E_{vv_*} = \{w \in \mathbb{R}^3 : w \cdot (v - v_*) = v \cdot (v - v_*)\}$ . By the use of Carleman's representation (see [12] and [8, Section 3]) we have,

$$\begin{aligned}(B^\varepsilon(t)f)(x, v) &= \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} f(x, v') g_t(x + \varepsilon\nu, \bar{v}')[(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \\ &= \int_{\mathbb{R}^3} k_t^\varepsilon(x, v, v_*) f(x, v_*) dv_*.\end{aligned}$$

**Lemma 3.3.18.** *For any  $f \in L^1_+(U \times \mathbb{R}^3)$  and any  $s, \varepsilon \geq 0$  we have  $\Sigma_t^\varepsilon U^\varepsilon(t, s)f \in L^1(U \times \mathbb{R}^3)$  for almost all  $t \geq s$ .*

*Proof.* We adapt the proof of [4, lemma 5.3]. Define  $Q : [s, \infty) \rightarrow [0, \infty)$  by,

$$\begin{aligned}Q(t) &:= \int_{U \times \mathbb{R}^3} \Sigma_t^\varepsilon(x, v) U^\varepsilon(t, s) f(x, v) dx dv \\ &= \int_{U \times \mathbb{R}^3} \Sigma_t^\varepsilon(x, v) \exp\left(-\int_s^t \Sigma_\sigma^\varepsilon(x - (t - \sigma)v, v) d\sigma\right) f(x - (t - s)v, v) dx dv\end{aligned}$$

Then for any  $r > s$  we have, using the substitution  $\bar{x} = x - tv$  and Fubini's theorem,

$$\begin{aligned}\int_s^r Q(t) dt &= \int_s^r \int_{U \times \mathbb{R}^3} \Sigma_t^\varepsilon(\bar{x} + tv, v) \exp\left(-\int_s^t \Sigma_\sigma^\varepsilon(\bar{x} + \sigma v) d\sigma\right) f(\bar{x} + sv, v) d\bar{x} dv dt \\ &= \int_{U \times \mathbb{R}^3} f(\bar{x} + sv, v) \int_s^r \Sigma_t^\varepsilon(\bar{x} + tv, v) \exp\left(-\int_s^t \Sigma_\sigma^\varepsilon(\bar{x} + \sigma v) d\sigma\right) dt d\bar{x} dv \\ &= \int_{U \times \mathbb{R}^3} f(\bar{x} + sv, v) \int_s^r -\partial_t \exp\left(-\int_s^t \Sigma_\sigma^\varepsilon(\bar{x} + \sigma v) d\sigma\right) dt d\bar{x} dv \\ &= \int_{U \times \mathbb{R}^3} f(\bar{x} + sv, v) \left(1 - \exp\left(-\int_s^r \Sigma_\sigma^\varepsilon(\bar{x} + \sigma v) d\sigma\right)\right) d\bar{x} dv \\ &= \|f\| - \|U(r, s)f\| < \infty.\end{aligned}$$

Hence  $Q(t)$  is finite for almost all  $t \geq s$  which proves the lemma.  $\square$

**Lemma 3.3.19.** *For any  $f \in L^1_+(U \times \mathbb{R}^3)$  and any  $s \geq 0$ ,  $U^\varepsilon(t, s)f \in D(B^\varepsilon(t))$  for almost every  $t \geq s$  and the mapping  $[s, \infty) \ni t \mapsto B^\varepsilon(t)U(t, s)f$  is measurable.*

Moreover, for any  $r \geq s$ ,

$$\int_s^r \|B^\varepsilon(t)U^\varepsilon(t,s)f\| dt = \int_s^r \|\Sigma_t^\varepsilon U^\varepsilon(t,s)f\| dt = \|f\| - \|U^\varepsilon(r,s)f\|.$$

*Proof.* By changing from pre to post collisional variables, see for example [19, Chapter 2, section 1.4.5], we have, for any  $x \in U$ ,

$$\begin{aligned} \int_{\mathbb{R}^3} B^\varepsilon(t)f(x,v) dv &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} f(x,v') g_t(x + \varepsilon\nu, \bar{v}') [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu dv \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} f(x,v) g_t(x + \varepsilon\nu, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu dv \\ &= \int_{\mathbb{R}^3} \Sigma_t^\varepsilon(x,v) f(x,v) dv. \end{aligned} \quad (3.3.39)$$

The required results now follow from the statement and proof of the previous lemma.  $\square$

*Proof of proposition 3.3.16.* For this proof we use [4]. The above two lemmas give that the modification of [4, lemma 5.11, corollary 5.12 and assumptions 5.1] to our situation hold. Hence, as in [4, section 5.2], we see that [4, theorem 2.1] holds, which gives that there exists an evolution family  $V^\varepsilon(t,s)$ . Hence  $f_t^\varepsilon := V^\varepsilon(t,0)f_0$  defines a solution to the non-autonomous linear Boltzmann equation (3.2.9). We now prove (3.3.38). We note that by [4, theorem 2.1] for any  $f \in L_+^1(U \times \mathbb{R}^3)$ ,

$$\int_{U \times \mathbb{R}^3} V^\varepsilon(t,0)f(x,v) dx dv \leq \int_{U \times \mathbb{R}^3} f(x,v) dx dv, \quad (3.3.40)$$

and,

$$f_t^\varepsilon = V^\varepsilon(t,0)f_0 = U^\varepsilon(t,0)f_0 + \int_0^t V^\varepsilon(t,r)B^\varepsilon(r)U^\varepsilon(r,0)f_0 dr.$$

Now by (3.2.3) for almost all  $x \in U$ ,

$$\begin{aligned} \Sigma_t^\varepsilon(x,v) &= \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x + \varepsilon\nu, \bar{v}) [(v - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \\ &\leq \pi \int_{\mathbb{R}^3} \bar{g}(\bar{v}) (|v| + |\bar{v}|) d\bar{v} \leq \pi M_g (1 + |v|). \end{aligned}$$

So by (3.3.39), noting that  $U^\varepsilon(r, 0)f_0(x, v)(1 + |v|) \in L_+^1(U \times \mathbb{R}^3)$ ,

$$\begin{aligned}
& \int_{U \times \mathbb{R}^3} B^\varepsilon(r) U^\varepsilon(r, 0) f_0(x, v) (1 + |v|) \, dx \, dv \\
&= \int_{U \times \mathbb{R}^3} \Sigma_t^\varepsilon(x, v) U^\varepsilon(r, 0) f_0(x, v) (1 + |v|) \, dx \, dv \\
&\leq \int_{U \times \mathbb{R}^3} \pi M_g (1 + |v|)^2 U^\varepsilon(r, 0) f_0(x, v) \, dx \, dv \\
&\leq 2\pi M_g \int_{U \times \mathbb{R}^3} f_0(x, v) (1 + |v|^2) \, dx \, dv.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \int_{U \times \mathbb{R}^3} f_t^\varepsilon(x, v) (1 + |v|) \, dx \, dv \\
&= \int_{U \times \mathbb{R}^3} U^\varepsilon(t, 0) f_0(x, v) (1 + |v|) \\
&\quad + \int_0^t V^\varepsilon(t, r) B^\varepsilon(r) U^\varepsilon(r, 0) f_0(x, v) (1 + |v|) \, dx \, dv \\
&\leq \int_{U \times \mathbb{R}^3} f_0(x, v) (1 + |v|) \, dx \, dv \\
&\quad + \int_0^t \int_{U \times \mathbb{R}^3} V^\varepsilon(t, r) B^\varepsilon(r) U^\varepsilon(r, 0) f_0(x, v) (1 + |v|) \, dx \, dv \, dr \\
&\leq \int_{U \times \mathbb{R}^3} f_0(x, v) (1 + |v|) \, dx \, dv + \int_0^t 2\pi M_g \int_{U \times \mathbb{R}^3} f_0(x, v) (1 + |v|^2) \, dx \, dv \, dr \\
&\leq M_f (1 + 2\pi T M_g) =: K < \infty.
\end{aligned}$$

□

**Remark 3.3.20.** We were unable to adapt the honesty results of [4] to our situation, so we cannot yet deduce that  $V^\varepsilon$  is an honest semigroup and that the solution  $f_t^\varepsilon = V^\varepsilon(t, 0)f_0$  conserves mass in the exact way. Honesty is proved later in proposition 3.3.37 by exploiting the connection to the idealised equation.

### 3.3.3 Building The Solution

In this subsection we construct the function  $P_t^\varepsilon(\Phi)$  iteratively and prove that it satisfies the properties in theorem 3.3.1. After defining  $P_t^\varepsilon(\Phi)$ , we define  $P_t^{\varepsilon, (j)}$ , which similarly to  $P_t^{\varepsilon, (0)}$ , can be thought of as the probability that the tagged particle is at a certain position and has experienced exactly  $j$  collisions. Once a few properties of  $P_t^{\varepsilon, (j)}$  have been checked the majority of theorem 3.3.1 follows. Proving that  $P_t^\varepsilon$  is

indeed a probability measure on  $\mathcal{MT}$  takes a number of technical lemmas.

This subsection differs from the previous chapter in two ways. Firstly the  $\varepsilon$  dependence, which makes little difference. Secondly there is a significant differences in proving that  $P_t^\varepsilon$  is a probability measure. In the previous chapter it followed from the honesty of the solution of the autonomous linear Boltzmann equation that the idealised distribution is a probability measure. However in this case we do not have the equivalent honesty result for the non-autonomous linear Boltzmann. Therefore we prove that  $P_t^\varepsilon$  is a probability measure by explicitly showing that the measure of the whole space has zero derivative with respect to time. This requires a significant number of calculations.

**Definition 3.3.21.** For  $j \in \mathbb{N} \cup \{0\}$  define,

$$\mathcal{T}_j := \{\Phi \in \mathcal{MT} : n(\Phi) = j\}. \quad (3.3.41)$$

That is,  $\mathcal{T}_j$  contains all histories with exactly  $j$  collisions. Let  $\varepsilon \geq 0$  and  $t \in [0, T]$ . For  $\Phi \in \mathcal{T}_0$  define

$$P_t^\varepsilon(\Phi) := P_t^{\varepsilon, (0)}(x(t), v(t)). \quad (3.3.42)$$

Else define,

$$\begin{aligned} P_t^\varepsilon(\Phi) := \mathbb{1}_{t \geq \tau} \exp \left( - \int_\tau^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x(\sigma) + \varepsilon\nu, \bar{v}) [(v(\tau) - \bar{v}) \cdot \nu']_+ d\bar{v} d\nu' d\sigma \right) \\ P_\tau^\varepsilon(\bar{\Phi}) g_\tau(x(\tau) + \varepsilon\nu, v') [(v(\tau^-) - v') \cdot \nu]_+. \end{aligned} \quad (3.3.43)$$

The right hand side of this equation depends on  $P_\tau^\varepsilon(\bar{\Phi})$  but since  $\bar{\Phi}$  has exactly one less collision than  $\Phi$  and we have defined  $P_t^\varepsilon(\Phi)$  for histories with zero collisions the equation is well defined. Note that this definition implies that for any,  $\varepsilon \geq 0$ ,  $\Phi \in \mathcal{MT}$  and  $\tau \leq s \leq t \leq T$ , with  $L_t^\varepsilon(\Phi)$  as defined in (3.3.5),

$$P_t^\varepsilon(\Phi) = \exp \left( - \int_s^t L_\sigma^\varepsilon(\Phi) d\sigma \right) P_s^\varepsilon(\Phi).$$

**Definition 3.3.22.** Let  $t \in [0, T]$ ,  $j \geq 1$ ,  $\varepsilon \geq 0$  and  $\Omega \subset U \times \mathbb{R}^3$  be measurable. Recall we define  $S_t(\Omega) = \{\Phi \in \mathcal{MT} : (x(t), v(t)) \in \Omega\}$ . Define,

$$S_t^j(\Omega) := S_t(\Omega) \cap \mathcal{T}_j. \quad (3.3.44)$$

Then define

$$P_t^{\varepsilon, (j)}(\Omega) := \int_{S_t^j(\Omega)} P_t^\varepsilon(\Phi) d\Phi. \quad (3.3.45)$$

**Lemma 3.3.23.** *Let  $t \in [0, T]$ ,  $j \geq 1$  and  $\varepsilon \geq 0$ . Then  $P_t^{\varepsilon, (j)}$  is absolutely continuous with respect to the Lebesgue measure on  $U \times \mathbb{R}^3$ .*

*Proof.* Let  $j = 1$  and  $\Omega \subset U \times \mathbb{R}^3$  be measurable. Then by (3.3.43) we have,

$$\begin{aligned} P_t^{\varepsilon, (1)}(\Omega) &= \int_{S_t^1(\Omega)} P_t^\varepsilon(\Phi) d\Phi \\ &= \int_0^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \int_U \int_{\mathbb{R}^3} \exp \left( - \int_\tau^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x(\sigma) + \varepsilon \nu', \bar{v}) \right. \\ &\quad \left. [(v(\tau) - \bar{v}) \cdot \nu']_+ d\bar{v} d\nu' d\sigma \right) P_\tau^{\varepsilon, (0)}(x_0 + \tau v_0, v_0) g_\tau(x_0 + \tau v_0 + \varepsilon \nu, v') \\ &\quad [(v_0 - v') \cdot \nu]_+ \mathbb{1}_{(x(t), v(t)) \in \Omega} dv_0 dx_0 dv' d\nu d\tau. \end{aligned} \quad (3.3.46)$$

We now introduce a change of coordinates  $(\nu, x_0, v_0, v') \mapsto (\nu, x, v, \bar{w})$  defined by,

$$\begin{aligned} v &= v_0 + \nu(v' - v_0) \cdot \nu \\ x &= x_0 + \tau v_0 + (t - \tau)v \\ \bar{w} &= v' - \nu(v' - v_0) \cdot \nu. \end{aligned}$$

Computing the Jacobian of this transformation,

$$\begin{pmatrix} \text{Id} & 0 & 0 & 0 \\ & \text{Id} & & \\ & 0 & \text{Id} - \nu \otimes \nu & \nu \otimes \nu \\ & 0 & \nu \otimes \nu & \text{Id} - \nu \otimes \nu \end{pmatrix}$$

where the non-filled entries are not required to compute the determinant. We now see that the bottom right 2x2 matrix has determinant  $-1$  and hence the absolute value of the determinant of the Jacobi matrix is 1. We note that under this transformation for  $t \geq \tau$ ,  $(x(t), v(t)) = (x, v)$  and for  $\tau \leq \sigma \leq t$ ,  $x(\sigma) = x - (t - \sigma)v$ . Hence with this



transformation (3.3.46) becomes,

$$\begin{aligned}
& P_t^{\varepsilon, (1)}(\Omega) \\
&= \int_{\Omega} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \exp \left( - \int_{\tau}^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_{\sigma}(x - (t - \sigma)v + \varepsilon\nu', \bar{v}) [(v - \bar{v}) \cdot \nu']_+ d\bar{v} d\nu' d\sigma \right) \\
&\quad P_{\tau}^{\varepsilon, (0)}(x - (t - \tau)v, w') g_{\tau}(x - (t - \tau)v + \varepsilon\nu, \bar{w}') \\
&\quad [(v - \bar{w}) \cdot \nu]_+ d\bar{w} d\nu d\tau dx dv,
\end{aligned} \tag{3.3.47}$$

where  $w' = v + \nu(\bar{w} - v) \cdot \nu$  and  $\bar{w}' = \bar{w} - \nu(\bar{w} - v) \cdot \nu$ . Hence we see that if the Lebesgue measure of  $\Omega$  is zero then  $P_t^{\varepsilon, (1)}(\Omega)$  equals zero also. For  $j \geq 1$  we use a similar approach using the iterative formula for  $P_t^{\varepsilon}(\Phi)$ .  $\square$

**Remark 3.3.24.** By the Radon-Nikodym theorem it follows that  $P_t^{\varepsilon, (j)}$  has a density, which we also denote by  $P_t^{\varepsilon, (j)}$ . Hence for any  $\Omega \subset U \times \mathbb{R}^3$  we have that,

$$\int_{\Omega} P_t^{\varepsilon, (j)}(x, v) dx dv = \int_{S_t^j(\Omega)} P_t^{\varepsilon}(\Phi) d\Phi,$$

This implies that for almost all  $(x, v) \in U \times \mathbb{R}^3$  we have,

$$P_t^{\varepsilon, (j)}(x, v) = \int_{S_t^j(x, v)} P_t^{\varepsilon}(\Phi) d\Phi.$$

**Proposition 3.3.25.** For any  $\varepsilon \geq 0$ ,  $j, t \geq 0$  for almost all  $(x, v) \in U \times \mathbb{R}^3$ ,

$$P_t^{\varepsilon, (j+1)}(x, v) = \int_0^t (U^{\varepsilon}(t, \tau) B^{\varepsilon}(\tau) P_{\tau}^{\varepsilon, (j)})(x, v) d\tau. \tag{3.3.48}$$

*Proof.* First consider  $j = 0$ . We prove that for any  $\Omega \subset U \times \mathbb{R}^3$  measurable,

$$\int_{\Omega} P_t^{\varepsilon, (1)}(x, v) dx dv = \int_{\Omega} \int_0^t (U^{\varepsilon}(t, \tau) B^{\varepsilon}(\tau) P_{\tau}^{\varepsilon, (0)})(x, v) d\tau dx dv. \tag{3.3.49}$$

By the definition of  $B^{\varepsilon}$  (3.3.10) and  $U^{\varepsilon}$  (3.3.12) we have, for  $w' = v + \nu(\bar{w} - v) \cdot \nu$  and

$$\bar{w}' = \bar{w} - \nu(\bar{w} - v) \cdot \nu,$$

$$\begin{aligned} & \int_{\Omega} \int_0^t (U^\varepsilon(t, \tau) B^\varepsilon(\tau) P_\tau^{\varepsilon, (0)})(x, v) \, d\tau \, dx \, dv \\ &= \int_{\Omega} \int_0^t U^\varepsilon(t, \tau) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} P_\tau^{\varepsilon, (0)}(x, w') g_\tau(x + \varepsilon \nu, \bar{w}') [(v - \bar{w}) \cdot \nu]_+ \, d\bar{w} \, d\nu \, d\tau \, dx \, dv \\ &= \int_{\Omega} \int_0^t \exp \left( - \int_\tau^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x + \varepsilon \nu' - (t - \sigma)v, \bar{v}) [(v - \bar{v}) \cdot \nu']_+ \, d\bar{v} \, d\nu' \, d\sigma \right) \\ & \quad \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} P_\tau^{\varepsilon, (0)}(x - (t - \tau)v, w') g_\tau(x - (t - \tau)v + \varepsilon \nu, \bar{w}') \\ & \quad [(v - \bar{w}) \cdot \nu]_+ \, d\bar{w} \, d\nu \, d\tau \, dx \, dv. \end{aligned}$$

Hence by (3.3.47) we notice that this is equal to the right hand side of (3.3.49). Hence for  $j = 0$  (3.3.48) holds for almost all  $(x, v) \in U \times \mathbb{R}^3$ . For  $j \geq 1$  one takes a similar approach. □

**Proposition 3.3.26.** *For almost all  $(x, v) \in U \times \mathbb{R}^3$ , for any  $\varepsilon, t \geq 0$ ,*

$$\sum_{j=0}^{\infty} P_t^{\varepsilon, (j)}(x, v) = f_t^\varepsilon(x, v).$$

*Proof.* In the proof of proposition 3.3.16 we saw, by using [4], that  $f_t^\varepsilon = V^\varepsilon(t, 0)f_0$ . By the proof of [4, theorem 2.1], we have,

$$f_t^\varepsilon = V^\varepsilon(t, 0)f_0 = \sum_{j=0}^{\infty} V_j^\varepsilon(t, 0)f_0, \quad (3.3.50)$$

where  $V_0^\varepsilon = U^\varepsilon$ , and for  $j \geq 0$ ,

$$V_{j+1}^\varepsilon(t, s) = \int_s^t V_j^\varepsilon(t, r) B^\varepsilon(r) U^\varepsilon(r, s) \, dr.$$

We notice by proposition 3.3.3,  $V_0^\varepsilon(t, 0)f_0 = U^\varepsilon(t, 0)f_0 = P_t^{\varepsilon, (0)}$ . Hence by proposition 3.3.25,

$$\begin{aligned} V_1^\varepsilon(t, 0)f_0 &= \int_0^t V_0^\varepsilon(t, r) B^\varepsilon(r) U^\varepsilon(r, 0)f_0 \, dr = \int_0^t U^\varepsilon(t, r) B^\varepsilon(r) U^\varepsilon(r, 0)f_0 \, dr \\ &= \int_0^t U^\varepsilon(t, r) B^\varepsilon(r) P_r^{\varepsilon, (0)} \, dr = P_t^{\varepsilon, (1)}. \end{aligned}$$

Further by Fubini's theorem and proposition 3.3.25,

$$\begin{aligned}
V_2^\varepsilon(t, 0)f_0 &= \int_0^t V_1^\varepsilon(t, r)B^\varepsilon(r)U^\varepsilon(r, 0)f_0 \, dr \\
&= \int_0^t \int_r^t U^\varepsilon(t, t_1)B^\varepsilon(t_1)U^\varepsilon(t_1, r)B^\varepsilon(r)U^\varepsilon(r, 0)f_0 \, dt_1 \, dr \\
&= \int_0^t \int_0^{t_1} U^\varepsilon(t, t_1)B^\varepsilon(t_1)U^\varepsilon(t_1, r)B^\varepsilon(r)U^\varepsilon(r, 0)f_0 \, dr \, dt_1 \\
&= \int_0^t U^\varepsilon(t, t_1)B^\varepsilon(t_1) \int_0^{t_1} U^\varepsilon(t_1, r)B^\varepsilon(r)U^\varepsilon(r, 0)f_0 \, dr \, dt_1 \\
&= \int_0^t U^\varepsilon(t, t_1)B^\varepsilon(t_1)P_{t_1}^{\varepsilon, (1)} \, dt_1 = P_t^{\varepsilon, (2)}.
\end{aligned}$$

Similarly we can see that for all  $j \geq 0$ ,  $V_j^\varepsilon(t, 0)f_0 = P_t^{\varepsilon, (j)}$ . Hence by (3.3.50) the required result holds.  $\square$

We can now prove the majority of theorem 3.3.1. The remainder of the proof requires a number of lemmas that follow.

**Proposition 3.3.27.** *For any  $\varepsilon \geq 0$  and  $t \in [0, T]$ ,  $P_t^\varepsilon \in L^1(\mathcal{MT})$  and  $P^\varepsilon$  is a solution to (3.3.2). Furthermore (3.3.6) and (3.3.7) hold.*

*Proof.* Firstly let  $\Omega \subset U \times \mathbb{R}^3$  be measurable. By proposition 3.3.26, definition 3.3.22 and since each  $P_t^{\varepsilon, (j)}$  is positive, the monotone convergence theorem,

$$\begin{aligned}
\int_\Omega f_t^\varepsilon(x, v) \, dx \, dv &= \int_\Omega \sum_{j=0}^\infty P_t^{\varepsilon, (j)}(x, v) \, dx \, dv = \sum_{j=0}^\infty \int_\Omega P_t^{\varepsilon, (j)}(x, v) \, dx \, dv \\
&= \sum_{j=0}^\infty \int_{S_t^j(\Omega)} P_t^\varepsilon(\Phi) \, d\Phi = \int_{S_t(\Omega)} P_t^\varepsilon(\Phi) \, d\Phi.
\end{aligned} \tag{3.3.51}$$

Hence for  $\Omega = U \times \mathbb{R}^3$  we have,

$$\int_{\mathcal{MT}} P_t^\varepsilon(\Phi) \, d\Phi = \int_{U \times \mathbb{R}^3} f_t^\varepsilon(x, v) \, dx \, dv < \infty. \tag{3.3.52}$$

Thus  $P_t^\varepsilon \in L^1(\mathcal{MT})$ . Now we check that  $P_t^\varepsilon(\Phi)$  indeed solves (3.3.2). For  $\Phi \in \mathcal{T}_0$  noting that  $x(t) = x_0 + tv_0$  and  $v(t) = v_0$ , we have for  $t \geq 0$

$$\begin{aligned}
P_t^\varepsilon(\Phi) &= P_t^{\varepsilon, (0)}(x(t), v(t)) = U^\varepsilon(t, 0)f_0(x(t), v(t)) \\
&= \exp \left( - \int_0^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x(\sigma) + \varepsilon\nu, \bar{v})[(v_0 - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu \, d\sigma \right) f_0(x_0, v_0).
\end{aligned} \tag{3.3.53}$$

We see that this gives the required initial value at  $t = 0$ , that it is differentiable with respect to  $t$  and differentiates to give the required term. Now consider  $\Phi \in \mathcal{T}_j$  for  $j \geq 1$ . By definition (3.3.43) we see that for  $t < \tau$ ,  $P_t^\varepsilon(\Phi) = 0$  and that  $P_\tau^\varepsilon(\Phi)$  has the required form. We also see that for  $t > \tau$  we have,

$$\begin{aligned}
\partial_t P_t^\varepsilon(\Phi) &= \partial_t \left( \exp \left( - \int_\tau^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x(\sigma) + \varepsilon\nu, \bar{v}) [(v(\tau) - \bar{v}) \cdot \nu']_+ d\bar{v} d\nu' d\sigma \right) \right. \\
&\quad \left. P_\tau(\bar{\Phi}) g_\tau(x(\tau) + \varepsilon\nu, v') [(v(\tau^-) - v') \cdot \nu]_+ \right) \\
&= - \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon\nu, \bar{v}) [(v(\tau) - \bar{v}) \cdot \nu']_+ d\bar{v} d\nu' \\
&\quad \exp \left( - \int_\tau^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x(\sigma) + \varepsilon\nu, \bar{v}) [(v(\tau) - \bar{v}) \cdot \nu']_+ d\bar{v} d\nu' d\sigma \right) \\
&\quad P_\tau(\bar{\Phi}) g_\tau(x(\tau) + \varepsilon\nu, v') [(v(\tau^-) - v') \cdot \nu]_+ \\
&= -L_t^\varepsilon(\Phi) P_t^\varepsilon(\Phi).
\end{aligned}$$

We now prove (3.3.6). Let  $K > 0$  be as in proposition 3.3.16. By a similar argument to (3.3.51), by using the same method as the proof of lemma 3.3.23 and by proposition 3.3.16 we have,

$$\begin{aligned}
\int_{\mathcal{MT}} P_t^\varepsilon(\Phi) (1 + |v(\tau)|) d\Phi &= \sum_{j=0}^{\infty} \int_{\mathcal{T}_j} P_t^\varepsilon(\Phi) (1 + |v(\tau)|) d\Phi \\
&= \sum_{j=0}^{\infty} \int_{U \times \mathbb{R}^3} P_t^{\varepsilon, (j)}(x, v) (1 + |v|) dx dv \\
&= \int_{U \times \mathbb{R}^3} \sum_{j=0}^{\infty} P_t^{\varepsilon, (j)}(x, v) (1 + |v|) dx dv = \int_{U \times \mathbb{R}^3} f_t^\varepsilon(x, v) (1 + |v|) dx dv \leq K.
\end{aligned}$$

We see that (3.3.7) has been proved in (3.3.51). □

The only remaining parts of theorem 3.3.1 are that  $P_t^\varepsilon$  is a probability measure on  $\mathcal{MT}$  and that (3.3.8) holds. We remark here that in chapter 2  $P_t$  being a probability measure resulted from the fact that we were able to prove that the semigroup defining the solution of the autonomous linear Boltzmann equation was honest and hence conserved mass. However in this non-autonomous case we have not been able to find equivalent honesty results. Therefore we prove that  $P_t^\varepsilon$  is a probability measure explicitly by showing that  $\int_{\mathcal{MT}} P_t^\varepsilon(\Phi) d\Phi$  is differentiable with respect to  $t$  and has derivative zero.

To that aim, the following lemmas calculate various limits that are required to show that  $\int_{\mathcal{MT}} P_t^\varepsilon(\Phi) d\Phi$  is differentiable, which is finally proved in lemma 3.3.33 and 3.3.34.

**Lemma 3.3.28.** *Let  $\varepsilon \geq 0$  and  $\Psi \in \mathcal{MT}$ . Then for  $t \geq \tau$ ,*

$$\begin{aligned} \frac{1}{h} \int_t^{t+h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} |P_s^\varepsilon(\Psi)g_s(x(s) + \varepsilon\bar{\nu}, \bar{v}) - P_t^\varepsilon(\Psi)g_t(x(t) + \varepsilon\bar{\nu}, \bar{v})| [(v(\tau) - \bar{v}) \cdot \bar{\nu}]_+ d\bar{v} d\bar{\nu} ds \\ \rightarrow 0 \text{ as } h \downarrow 0. \end{aligned} \quad (3.3.54)$$

And for  $t > \tau$

$$\begin{aligned} \frac{1}{h} \int_{t-h}^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} |P_s^\varepsilon(\Psi)g_s(x(s) + \varepsilon\bar{\nu}, \bar{v}) - P_t^\varepsilon(\Psi)g_t(x(t) + \varepsilon\bar{\nu}, \bar{v})| [(v(\tau) - \bar{v}) \cdot \bar{\nu}]_+ d\bar{v} d\bar{\nu} ds \\ \rightarrow 0 \text{ as } h \downarrow 0. \end{aligned} \quad (3.3.55)$$

*Proof.* We begin with (3.3.54). Let  $t \geq \tau$ . Firstly,

$$\begin{aligned} |P_s^\varepsilon(\Psi)g_s(x(s) + \varepsilon\bar{\nu}, \bar{v}) - P_t^\varepsilon(\Psi)g_t(x(t) + \varepsilon\bar{\nu}, \bar{v})| \\ \leq P_t^\varepsilon(\Psi)|g_s(x(s) + \varepsilon\bar{\nu}, \bar{v}) - g_t(x(t) + \varepsilon\bar{\nu}, \bar{v})| + g_s(x(s) + \varepsilon\bar{\nu}, \bar{v})|P_t^\varepsilon(\Psi) - P_s^\varepsilon(\Psi)|. \end{aligned} \quad (3.3.56)$$

Noting that for  $s, t \geq \tau$ ,  $|x(s) - x(t)| = |t - s||v(\tau)|$  it follows by lemma 3.3.9, with  $R = h^{-\alpha/6}$  that,

$$\begin{aligned} \frac{1}{h} \int_t^{t+h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} P_t^\varepsilon(\Psi)|g_s(x(s) + \varepsilon\bar{\nu}, \bar{v}) - g_t(x(t) + \varepsilon\bar{\nu}, \bar{v})| [(v(\tau) - \bar{v}) \cdot \bar{\nu}]_+ d\bar{v} d\bar{\nu} ds \\ \leq \frac{1}{h} \int_t^{t+h} CP_t^\varepsilon(\Psi)(1 + |v(\tau)|) \left( h^{\alpha/6} + h^{-5\alpha/6}|t - s|^\alpha(1 + |v(\tau)|^\alpha) \right) ds \\ = CP_t^\varepsilon(\Psi)(1 + |v(\tau)|) \left( h^{\alpha/6} + h^{-5\alpha/6}(1 + |v(\tau)|^\alpha) \frac{1}{h} \int_{t-h}^t |t - s|^\alpha ds \right) \\ = CP_t^\varepsilon(\Psi)(1 + |v(\tau)|) \left( h^{\alpha/6} + h^{-5\alpha/6}(1 + |v(\tau)|^\alpha) \frac{h^\alpha}{\alpha + 1} \right) \\ = CP_t^\varepsilon(\Psi)(1 + |v(\tau)|) \left( 1 + \frac{1 + |v(\tau)|^\alpha}{\alpha + 1} \right) h^{\alpha/6} \\ \rightarrow 0 \text{ as } h \rightarrow 0. \end{aligned} \quad (3.3.57)$$

Now let  $\delta > 0$ . By the proof of proposition 3.3.27 we know that  $P_s^\varepsilon(\Psi)$  is differentiable with respect to  $s$  and hence continuous for  $s \geq \tau$ . So for  $h$  sufficiently small and any

$s \in [t, t+h]$ ,

$$|P_t^\varepsilon(\Psi) - P_s^\varepsilon(\Psi)| < \frac{\delta}{\pi M_g(1 + |v(\tau)|)}.$$

Thus,

$$\begin{aligned} & \frac{1}{h} \int_t^{t+h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_s(x(s) + \varepsilon \bar{\nu}, \bar{v}) |P_t^\varepsilon(\Psi) - P_s^\varepsilon(\Psi)| [(v(\tau) - \bar{v}) \cdot \bar{\nu}]_+ d\bar{v} d\bar{\nu} ds \\ & < \frac{\delta}{\pi M_g(1 + |v(\tau)|)} \frac{1}{h} \int_t^{t+h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_s(x(s) + \varepsilon \bar{\nu}, \bar{v}) [(v(\tau) - \bar{v}) \cdot \bar{\nu}]_+ d\bar{v} d\bar{\nu} ds \\ & < \frac{\delta}{\pi M_g(1 + |v(\tau)|)} \frac{1}{h} \int_t^{t+h} \pi M_g(1 + |v(\tau)|) ds = \delta. \end{aligned}$$

This, together with (3.3.56) and (3.3.57) proves (3.3.54).

The proof of (3.3.55) is similar but we must exclude  $t = \tau$  because in that case  $P_s^\varepsilon(\Psi)$  in the integrand is always 0. For  $t > \tau$  we take  $h$  sufficiently small so that  $t - h > \tau$  and hence  $P_s^\varepsilon(\Psi)$  is continuous with respect to  $s$  for  $s \in [t - h, t]$ . The result now follows by the same method as (3.3.54).  $\square$

**Definition 3.3.29.** For any  $S \subset [0, T]$  measurable define,  $\mathcal{MT}_S$  by,

$$\mathcal{MT}_S := \{\Phi \in \mathcal{MT} : \tau \in S\}. \quad (3.3.58)$$

And for  $t \in [0, T]$  define

$$\mathcal{MT}_t := \mathcal{MT}_{\{t\}} = \{\Phi \in \mathcal{MT} : \tau = t\}. \quad (3.3.59)$$

**Lemma 3.3.30.** For any  $t \in (0, T]$ ,  $\mathcal{MT}_t$  is a set of zero measure with respect to the Lebesgue measure on  $\mathcal{MT}$ .

*Proof.* Let  $t \in (0, T]$ . Then  $\mathcal{MT}_t \cap \mathcal{T}_0 = \emptyset$  since for any  $\Phi \in \mathcal{T}_0$ ,  $\tau = 0$ . Now for any  $j \geq 1$ ,  $\mathcal{MT}_t \cap \mathcal{T}_j$  is a set of co-dimension 1 in  $\mathcal{T}_j$  (since one component, the final collision time, is fixed) and hence has zero measure. Since,

$$\mathcal{MT}_t = \cup_{j \geq 1} \mathcal{MT}_t \cap \mathcal{T}_j$$

it follows that  $\mathcal{MT}_t$  is set of zero measure.  $\square$

**Definition 3.3.31.** Let  $\Psi \in \mathcal{MT}$ . For  $s \in (\tau, T]$ ,  $\bar{\nu} \in \mathbb{S}^2$  and  $\bar{v} \in \mathbb{R}^3$ , when the context is clear let  $\Psi' := \Psi \cup (s, \bar{\nu}, \bar{v})$  denote the collision history formed by adding the collision  $(s, \bar{\nu}, \bar{v})$  to  $\Psi$ .

**Lemma 3.3.32.** For  $t \in [0, T]$ ,  $\varepsilon \geq 0$  and any  $\Psi \in \mathcal{MT}$ ,

$$\begin{aligned} L_t^\varepsilon(\Psi)P_t^\varepsilon(\Psi) &= \lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \exp\left(-\int_s^{t+h} L_\sigma^\varepsilon(\Psi') d\sigma\right) P_s^\varepsilon(\Psi) g_s(x(s) + \varepsilon \bar{\nu}, \bar{v}) \\ &\quad [(v(\tau) - \bar{v}) \cdot \bar{\nu}]_+ d\bar{v} d\bar{\nu} ds. \end{aligned} \quad (3.3.60)$$

For  $t \in (0, T]$ ,  $\varepsilon \geq 0$  and for almost all  $\Psi \in \mathcal{MT}$ ,

$$\begin{aligned} L_t^\varepsilon(\Psi)P_t^\varepsilon(\Psi) &= \lim_{h \downarrow 0} \frac{1}{h} \int_{t-h}^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \exp\left(-\int_s^t L_\sigma^\varepsilon(\Psi') d\sigma\right) P_s^\varepsilon(\Psi) g_s(x(s) + \varepsilon \bar{\nu}, \bar{v}) \\ &\quad [(v(\tau) - \bar{v}) \cdot \bar{\nu}]_+ d\bar{v} d\bar{\nu} ds. \end{aligned} \quad (3.3.61)$$

*Proof.* Let  $t \in [0, T]$ . We first prove (3.3.60). Let  $\Psi \in \mathcal{MT}$ . If  $t < \tau$  then the left hand side is zero and the right side is zero also, since for  $h$  sufficiently small  $P_s^\varepsilon(\Psi) = 0$  for all  $s \in [t, t+h]$ . Suppose  $t \geq \tau$ . Note that,

$$\begin{aligned} L_t^\varepsilon(\Psi)P_t^\varepsilon(\Psi) &= \frac{1}{h} \int_t^{t+h} L_t^\varepsilon(\Psi)P_t^\varepsilon(\Psi) ds \\ &= \frac{1}{h} \int_t^{t+h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} P_t^\varepsilon(\Psi) g_t(x(t) + \varepsilon \bar{\nu}, \bar{v}) [(v(\tau) - \bar{v}) \cdot \bar{\nu}]_+ d\bar{v} d\bar{\nu} ds. \end{aligned}$$

Hence to prove (3.3.60) we show that,

$$\begin{aligned} &\frac{1}{h} \int_t^{t+h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \left| \exp\left(-\int_s^{t+h} L_\sigma^\varepsilon(\Psi') d\sigma\right) P_s^\varepsilon(\Psi) g_s(x(s) + \varepsilon \bar{\nu}, \bar{v}) - P_t^\varepsilon(\Psi) g_t(x(t) + \varepsilon \bar{\nu}, \bar{v}) \right| \\ &\quad [(v(\tau) - \bar{v}) \cdot \bar{\nu}]_+ d\bar{v} d\bar{\nu} ds \\ &\rightarrow 0 \text{ as } h \downarrow 0. \end{aligned}$$

Now,

$$\begin{aligned} &\left| \exp\left(-\int_s^{t+h} L_\sigma^\varepsilon(\Psi') d\sigma\right) P_s^\varepsilon(\Psi) g_s(x(s) + \varepsilon \bar{\nu}, \bar{v}) - P_t^\varepsilon(\Psi) g_t(x(t) + \varepsilon \bar{\nu}, \bar{v}) \right| \\ &\leq |P_s^\varepsilon(\Psi) g_s(x(s) + \varepsilon \bar{\nu}, \bar{v}) - P_t^\varepsilon(\Psi) g_t(x(t) + \varepsilon \bar{\nu}, \bar{v})| \\ &\quad + \left(1 - \exp\left(-\int_s^{t+h} L_\sigma^\varepsilon(\Psi') d\sigma\right)\right) P_t^\varepsilon(\Psi) g_t(x(t) + \varepsilon \bar{\nu}, \bar{v}). \end{aligned}$$

So by using (3.3.54) it remains to prove that,

$$\begin{aligned}
I(h) &:= \frac{1}{h} \int_t^{t+h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \left( 1 - \exp \left( - \int_s^{t+h} L_\sigma^\varepsilon(\Psi') d\sigma \right) \right) P_t^\varepsilon(\Psi) \\
&\quad g_t(x(t) + \varepsilon \bar{\nu}, \bar{v}) [(v(\tau) - \bar{v}) \cdot \bar{\nu}]_+ d\bar{v} d\bar{\nu} ds \\
&\rightarrow 0 \text{ as } h \downarrow 0.
\end{aligned} \tag{3.3.62}$$

Recall  $\Psi' = \Psi \cup (s, \bar{\nu}, \bar{v})$ . Denote by  $w$  the velocity of the tagged particle of  $\Psi'$  after its final collision at  $s$ . Then,

$$w = v(\tau) + \bar{\nu}(v(\tau) - \bar{v}) \cdot \bar{\nu}$$

Hence,

$$|w| \leq |v(\tau)| + |v(\tau) - \bar{v}| \leq 2|v(\tau)| + |\bar{v}|.$$

Thus for  $s \in [t, t+h]$ ,

$$\begin{aligned}
\int_s^{t+h} L_\sigma^\varepsilon(\Psi') d\sigma &= \int_s^{t+h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_\sigma(x(\Psi')(\sigma) + \varepsilon \nu_1, v_1) [(w - v_1) \cdot \nu_1]_+ d\nu_1 d\nu_1 d\sigma \\
&\leq \pi \int_s^{t+h} \int_{\mathbb{R}^3} \bar{g}(v_1) (|w| + |v_1|) dv_1 d\sigma \\
&\leq \pi M_g (1 + |w|) (t + h - s) \\
&\leq h\pi M_g (1 + 2|v(\tau)| + |\bar{v}|).
\end{aligned}$$

It follows that,

$$1 - \exp \left( - \int_s^{t+h} L_\sigma^\varepsilon(\Psi') d\sigma \right) \leq 1 - \exp \left( - h\pi M_g (1 + 2|v(\tau)| + |\bar{v}|) \right).$$

Hence,

$$\begin{aligned}
I(h) &\leq \frac{1}{h} \int_t^{t+h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} (1 - \exp \left( - h\pi M_g (1 + 2|v(\tau)| + |\bar{v}|) \right)) \\
&\quad P_t^\varepsilon(\Psi) \bar{g}(\bar{v}) (|v(\tau)| + |\bar{v}|) d\bar{v} d\bar{\nu} ds \\
&\leq \pi P_t^\varepsilon(\Psi) \int_{\mathbb{R}^3} (1 - \exp \left( - h\pi M_g (1 + 2|v(\tau)| + |\bar{v}|) \right)) \bar{g}(\bar{v}) (|v(\tau)| + |\bar{v}|) d\bar{v}.
\end{aligned} \tag{3.3.63}$$

Let  $\delta > 0$ . By (3.2.3) there exists an  $R > 0$  such that,

$$\int_{\mathbb{R}^3 \setminus B_R(0)} \bar{g}(\bar{v}) (1 + |\bar{v}|) d\bar{v} < \frac{\delta}{\pi(1 + P_t^\varepsilon(\Psi))(1 + |v(\tau)|)}.$$



Hence,

$$\begin{aligned}
& \pi P_t^\varepsilon(\Psi) \int_{\mathbb{R}^3 \setminus B_R(0)} (1 - \exp(-h\pi M_g(1 + 2|v(\tau)| + |\bar{v}|))) \bar{g}(\bar{v})(|v(\tau)| + |\bar{v}|) d\bar{v} \\
& \leq \pi P_t^\varepsilon(\Psi) \int_{\mathbb{R}^3 \setminus B_R(0)} \bar{g}(\bar{v})(|v(\tau)| + |\bar{v}|) d\bar{v} \\
& < \delta.
\end{aligned} \tag{3.3.64}$$

Further for  $h$  sufficiently small,

$$\begin{aligned}
& \pi P_t^\varepsilon(\Psi) \int_{B_R(0)} (1 - \exp(-h\pi M_g(1 + 2|v(\tau)| + |\bar{v}|))) \bar{g}(\bar{v})(|v(\tau)| + |\bar{v}|) d\bar{v} \\
& \leq \pi P_t^\varepsilon(\Psi) (1 - \exp(-h\pi M_g(1 + 2|v(\tau)| + R))) \int_{B_R(0)} \bar{g}(\bar{v})(|v(\tau)| + R) d\bar{v} \\
& \leq \pi M_g P_t^\varepsilon(\Psi)(|v(\tau)| + R) (1 - \exp(-h\pi M_g(1 + 2|v(\tau)| + R))) \\
& < \delta.
\end{aligned}$$

By substituting this and (3.3.64) into (3.3.63) we see that (3.3.62) holds, which concludes the proof of (3.3.60).

We now prove (3.3.61), which we prove holds for all  $\Psi \in \mathcal{MT} \setminus \mathcal{MT}_t$ . Indeed  $\mathcal{MT}_t$  is a set of zero measure by lemma 3.3.30. Let  $\Psi \in \mathcal{MT} \setminus \mathcal{MT}_t$ . If  $\tau > t$  then the left hand side of (3.3.61) is zero and the right hand side side is also zero since for any  $s \in [t - h, t]$ ,  $P_s^\varepsilon(\Phi) = 0$ . If  $t > \tau$  we use the same method as we used for (3.3.60), using (3.3.55) instead of (3.3.54).  $\square$

**Lemma 3.3.33.** *Let  $\varepsilon \geq 0$  and  $t \in [0, T]$ . Then  $\partial_t^+ \int_{\mathcal{MT}} P_t^\varepsilon(\Phi) d\Phi$  exists and is equal to zero.*

*Proof.* Fix  $\varepsilon \geq 0$  and  $t \in [0, T]$ . We want to show that

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{\mathcal{MT}} P_{t+h}^\varepsilon(\Phi) - P_t^\varepsilon(\Phi) d\Phi = 0.$$

For  $\mathcal{MT}_S$  as defined in definition 3.3.29 and  $h > 0$  we have,

$$\begin{aligned}
& \frac{1}{h} \int_{\mathcal{MT}} P_{t+h}^\varepsilon(\Phi) - P_t^\varepsilon(\Phi) d\Phi \\
& = \frac{1}{h} \int_{\mathcal{MT}_{[0,t]}} P_{t+h}^\varepsilon(\Phi) - P_t^\varepsilon(\Phi) d\Phi + \frac{1}{h} \int_{\mathcal{MT}_{(t,t+h]}} P_{t+h}^\varepsilon(\Phi) - P_t^\varepsilon(\Phi) d\Phi \\
& \quad + \frac{1}{h} \int_{\mathcal{MT}_{(t+h,T]}} P_{t+h}^\varepsilon(\Phi) - P_t^\varepsilon(\Phi) d\Phi.
\end{aligned}$$

We remark here that in the case  $t = 0$ ,  $\mathcal{MT}_0$  is the set of all histories with zero collisions and is not a set of zero measure in  $\mathcal{MT}$  since it is equivalent to  $U \times \mathbb{R}^3$ . We show that each of these terms converges and that their sum is zero. Firstly,

$$\frac{1}{h} \int_{\mathcal{MT}_{(t+h, T]}} P_{t+h}^\varepsilon(\Phi) - P_t^\varepsilon(\Phi) \, d\Phi = \frac{1}{h} \int_{\mathcal{MT}_{(t+h, T]}} 0 \, d\Phi = 0. \quad (3.3.65)$$

Now note that for any  $h > 0$ ,  $\int_t^{t+h} L_s^\varepsilon(\Phi) \, ds \leq h\pi M_g(1 + |v(\tau)|)$ . Hence for any  $h > 0$ ,

$$\begin{aligned} \frac{1}{h} \left| \exp \left( - \int_t^{t+h} L_s^\varepsilon(\Phi) \, ds \right) - 1 \right| &\leq \frac{1}{h} (1 - \exp(-h\pi M_g(1 + |v(\tau)|))) \\ &\leq \frac{1}{h} (1 - 1 - h\pi M_g(1 + |v(\tau)|)) = \pi M_g(1 + |v(\tau)|). \end{aligned}$$

By (3.3.6) it follows that,

$$\begin{aligned} \int_{\mathcal{MT}_{[0, t]}} \frac{1}{h} \left| \exp \left( - \int_t^{t+h} L_s^\varepsilon(\Phi) \, ds \right) - 1 \right| P_t^\varepsilon(\Phi) \, d\Phi \\ \leq \int_{\mathcal{MT}} \pi M_g(1 + |v(\tau)|) P_t^\varepsilon(\Phi) \, d\Phi \leq \pi M_g K < \infty. \end{aligned}$$

Hence by the dominated convergence theorem and the fact that for any  $\Phi$  with  $\tau > t$ ,  $P_t^\varepsilon(\Phi) = 0$ ,

$$\begin{aligned} \frac{1}{h} \int_{\mathcal{MT}_{[0, t]}} P_{t+h}^\varepsilon(\Phi) - P_t^\varepsilon(\Phi) \, d\Phi \\ = \frac{1}{h} \int_{\mathcal{MT}_{[0, t]}} \exp \left( - \int_t^{t+h} L_s^\varepsilon(\Phi) \, ds \right) P_t^\varepsilon(\Phi) - P_t^\varepsilon(\Phi) \, d\Phi \\ = \int_{\mathcal{MT}_{[0, t]}} \frac{1}{h} \left( \exp \left( - \int_t^{t+h} L_s^\varepsilon(\Phi) \, ds \right) - 1 \right) P_t^\varepsilon(\Phi) \, d\Phi \\ \xrightarrow{h \downarrow 0} \int_{\mathcal{MT}_{[0, t]}} \partial_\sigma|_{\sigma=t} \exp \left( - \int_t^\sigma L_s^\varepsilon(\Phi) \, ds \right) P_t^\varepsilon(\Phi) \, d\Phi \\ = - \int_{\mathcal{MT}} L_t^\varepsilon(\Phi) P_t^\varepsilon(\Phi) \, d\Phi. \end{aligned} \quad (3.3.66)$$

Now,

$$\begin{aligned}
\frac{1}{h} \int_{\mathcal{MT}_{(t,t+h]}} P_{t+h}^\varepsilon(\Phi) - P_t^\varepsilon(\Phi) d\Phi &= \frac{1}{h} \int_{\mathcal{MT}_{(t,t+h]}} P_{t+h}^\varepsilon(\Phi) d\Phi \\
&= \frac{1}{h} \int_{\mathcal{MT}_{(t,t+h]}} \exp\left(-\int_\tau^{t+h} L_\sigma^\varepsilon(\Phi) d\sigma\right) P_\tau^\varepsilon(\Phi) d\Phi \\
&= \frac{1}{h} \int_{\mathcal{MT}_{(t,t+h]}} \exp\left(-\int_\tau^{t+h} L_\sigma^\varepsilon(\Phi) d\sigma\right) P_\tau^\varepsilon(\bar{\Phi}) g_\tau(x(\tau) + \varepsilon\nu, v') [(v(\tau^-) - v') \cdot \nu]_+ d\Phi \\
&= \int_{\mathcal{MT}} \frac{1}{h} \int_t^{t+h} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \exp\left(-\int_s^{t+h} L_\sigma^\varepsilon(\Psi') d\sigma\right) P_s^\varepsilon(\Psi) \\
&\quad g_s(x(s) + \varepsilon\nu, \bar{v}) [(v(\tau) - \bar{v}) \cdot \bar{\nu}]_+ d\bar{v} d\bar{\nu} ds d\Psi.
\end{aligned}$$

Hence by the dominated convergence theorem and (3.3.60),

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{\mathcal{MT}_{(t,t+h]}} P_{t+h}^\varepsilon(\Phi) - P_t^\varepsilon(\Phi) d\Phi = \int_{\mathcal{MT}} L_t^\varepsilon(\Phi) P_t^\varepsilon(\Phi). \quad (3.3.67)$$

Combining (3.3.65), (3.3.66) and (3.3.67) we see that the limit indeed exists and is equal to zero, proving the lemma.  $\square$

**Lemma 3.3.34.** *Let  $\varepsilon \geq 0$  and  $t \in (0, T]$ . Then  $\partial_t^- \int_{\mathcal{MT}} P_t^\varepsilon(\Phi)$  exists and is equal to zero.*

*Proof.* Fix  $\varepsilon \geq 0$  and  $t \in (0, T]$ . We show that,

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{\mathcal{MT}} P_t^\varepsilon(\Phi) - P_{t-h}^\varepsilon(\Phi) d\Phi = 0.$$

As in lemma 3.3.33 note that,

$$\begin{aligned}
&\frac{1}{h} \int_{\mathcal{MT}} P_t^\varepsilon(\Phi) - P_{t-h}^\varepsilon(\Phi) d\Phi \\
&= \frac{1}{h} \int_{\mathcal{MT}_{[0,t-h]}} P_t^\varepsilon(\Phi) - P_{t-h}^\varepsilon(\Phi) d\Phi + \frac{1}{h} \int_{\mathcal{MT}_{(t-h,t]}} P_t^\varepsilon(\Phi) - P_{t-h}^\varepsilon(\Phi) d\Phi \\
&\quad + \frac{1}{h} \int_{\mathcal{MT}_{(t,T]}} P_t^\varepsilon(\Phi) - P_{t-h}^\varepsilon(\Phi) d\Phi.
\end{aligned}$$

We again show each limit exists and the sum is zero. Firstly,

$$\frac{1}{h} \int_{\mathcal{MT}_{(t,T]}} P_t^\varepsilon(\Phi) - P_{t-h}^\varepsilon(\Phi) d\Phi = 0. \quad (3.3.68)$$

By lemma 3.3.30, (3.3.61) and the dominated convergence theorem we have,

$$\begin{aligned}
& \frac{1}{h} \int_{\mathcal{MT}_{(t-h,t]}} P_t^\varepsilon(\Phi) - P_{t-h}^\varepsilon(\Phi) d\Phi = \frac{1}{h} \int_{\mathcal{MT}_{(t-h,t)}} P_t^\varepsilon(\Phi) d\Phi \\
& = \frac{1}{h} \int_{\mathcal{MT}_{(t-h,t)}} \exp\left(-\int_\tau^t L_\sigma^\varepsilon(\Phi) d\sigma\right) P_\tau^\varepsilon(\bar{\Phi}) g_\tau(x(\tau) + \varepsilon\nu, v') [(v(\tau^-) - v') \cdot \nu]_+ d\Phi \\
& = \int_{\mathcal{MT}} \frac{1}{h} \int_{t-h}^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \exp\left(-\int_s^t L_\sigma^\varepsilon(\Psi') d\sigma\right) P_s^\varepsilon(\Psi) \\
& \quad g_s(x(s) + \varepsilon\bar{\nu}, \bar{v}) [(v(\tau) - \bar{v}) \cdot \bar{\nu}]_+ d\bar{\nu} d\bar{\nu} ds d\Psi \\
& \xrightarrow{h \downarrow 0} \int_{\mathcal{MT}} L_t^\varepsilon(\Psi) P_t^\varepsilon(\Psi) d\Psi. \tag{3.3.69}
\end{aligned}$$

Now for the final term we first prove that for any  $\Phi \in \mathcal{MT}_{[0,t]}$ ,

$$\lim_{h \downarrow 0} \frac{1}{h} P_{t-h}^\varepsilon(\Phi) \left( \exp\left(-\int_{t-h}^t L_\sigma^\varepsilon(\Phi) d\sigma\right) - 1 \right) = -L_t^\varepsilon(\Phi) P_t^\varepsilon(\Phi). \tag{3.3.70}$$

To this aim fix  $\Phi \in \mathcal{MT}_{[0,t]}$ . Then  $\tau < t$ . Let  $h$  sufficiently small so that  $t-h > \tau$ . Then since  $P_s^\varepsilon(\Phi)$  is continuous for  $s \in [\tau, T]$  we have that  $P_{t-h}^\varepsilon(\Phi)$  converges to  $P_t^\varepsilon(\Phi)$  as  $h$  tends to zero. Further,

$$\lim_{h \downarrow 0} \frac{1}{h} \left( \exp\left(-\int_{t-h}^t L_\sigma^\varepsilon(\Phi) d\sigma\right) - 1 \right) = -\partial_s|_{s=t} \exp\left(-\int_s^t L_\sigma^\varepsilon(\Phi) d\sigma\right) = -L_t^\varepsilon(\Phi).$$

This proves (3.3.70). Hence by the dominated convergence theorem and lemma 3.3.30,

$$\begin{aligned}
& \frac{1}{h} \int_{\mathcal{MT}_{[0,t-h]}} P_t^\varepsilon(\Phi) - P_{t-h}^\varepsilon(\Phi) d\Phi \\
& = \frac{1}{h} \int_{\mathcal{MT}_{[0,t-h]}} \exp\left(-\int_{t-h}^t L_\sigma^\varepsilon(\Phi) d\sigma\right) P_{t-h}^\varepsilon(\Phi) - P_{t-h}^\varepsilon(\Phi) d\Phi \\
& = \int_{\mathcal{MT}_{[0,t]}} \frac{1}{h} P_{t-h}^\varepsilon(\Phi) \left( \exp\left(-\int_{t-h}^t L_\sigma^\varepsilon(\Phi) d\sigma\right) - 1 \right) d\Phi \\
& \xrightarrow{h \downarrow 0} \int_{\mathcal{MT}_{[0,t]}} -L_t^\varepsilon(\Phi) P_t^\varepsilon(\Phi) d\Phi \\
& = \int_{\mathcal{MT}_{[0,t]}} -L_t^\varepsilon(\Phi) P_t^\varepsilon(\Phi) d\Phi \\
& = \int_{\mathcal{MT}} -L_t^\varepsilon(\Phi) P_t^\varepsilon(\Phi) d\Phi. \tag{3.3.71}
\end{aligned}$$

Combining (3.3.68), (3.3.69) and (3.3.71) proves the lemma.  $\square$

The following lemmas are used to prove (3.3.8).

**Lemma 3.3.35.** For  $\varepsilon > 0$  sufficiently small, almost all  $\Phi \in \mathcal{MT}$  and any  $t \in [0, T]$ ,

$$|L_t^0(\Phi) - L_t^\varepsilon(\Phi)| \leq 2C(1 + |v(\tau)|)\varepsilon^{\alpha/6},$$

where  $C$  is as in lemma 3.3.9.

*Proof.* Let  $R = \varepsilon^{-\alpha/6}$  and  $\varepsilon$  sufficiently small so that  $R \geq 1$ . Let  $\Phi \in \mathcal{MT}$  be such that for all  $t \in [0, T]$ ,  $\nu \in \mathbb{S}^2$ , and almost all  $\bar{v} \in \mathbb{R}^3$ ,

$$\begin{aligned} g_t(x(t), \bar{v}) + g_t(x(t) + \varepsilon\nu, \bar{v}) &\leq 2\bar{g}(\bar{v}) \text{ and,} \\ |g_t(x(t), \bar{v}) - g_t(x(t) + \varepsilon\nu, \bar{v})| &\leq M\varepsilon^\alpha. \end{aligned}$$

Indeed by (3.2.2) and (3.2.6) this only excludes a set of zero measure. As in the proof of lemma 3.3.9,

$$\begin{aligned} &\int_{\mathbb{S}^2} \int_{\mathbb{R}^3 \setminus B_R(0)} |g_t(x(t), \bar{v}) - g_t(x(t) + \varepsilon\nu, \bar{v})| [(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \\ &\leq \int_{\mathbb{S}^2} \int_{\mathbb{R}^3 \setminus B_R(0)} (g_t(x(t), \bar{v}) + g_t(x(t) + \varepsilon\nu, \bar{v})) (|v(\tau)| + |\bar{v}|) d\bar{v} d\nu \\ &\leq \int_{\mathbb{S}^2} \int_{\mathbb{R}^3 \setminus B_R(0)} 2\bar{g}(\bar{v}) (|v(\tau)| + |\bar{v}|) d\bar{v} d\nu \leq 4\pi \frac{M_g}{R} (1 + |v(\tau)|). \end{aligned} \quad (3.3.72)$$

And

$$\begin{aligned} &\int_{\mathbb{S}^2} \int_{B_R(0)} |g_t(x(t), \bar{v}) - g_t(x(t) + \varepsilon\nu, \bar{v})| [(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \\ &\leq \int_{\mathbb{S}^2} \int_{B_R(0)} M\varepsilon^\alpha (|v(\tau)| + |\bar{v}|) d\bar{v} d\nu \\ &\leq \int_{\mathbb{S}^2} \int_{B_R(0)} M\varepsilon^\alpha (|v(\tau)| + R) d\bar{v} d\nu \\ &\leq \frac{4}{3}\pi R^3 \times 2\pi \times M\varepsilon^\alpha \times R(1 + |v(\tau)|) \\ &\leq \frac{8}{3}\pi^2 M\varepsilon^\alpha R^5 (1 + |v(\tau)|). \end{aligned} \quad (3.3.73)$$

Combining (3.3.72) and (3.3.73) we have for  $C$  as in lemma 3.3.9,

$$\begin{aligned} |L_t^0(\Phi) - L_t^\varepsilon(\Phi)| &\leq \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} |g_t(x(t), \bar{v}) - g_t(x(t) + \varepsilon\nu, \bar{v})| [(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu \\ &\leq 4\pi \frac{M_g}{R} (1 + |v(\tau)|) + \frac{8}{3}\pi^2 M\varepsilon^\alpha R^5 (1 + |v(\tau)|) \\ &\leq C(1 + |v(\tau)|) \left( \frac{1}{R} + \varepsilon^\alpha R^5 \right). \end{aligned}$$

Substituting  $R = \varepsilon^{-\alpha/6}$  gives the required result.  $\square$

**Lemma 3.3.36.** *For almost all  $\Phi \in \mathcal{MT}$ , uniformly for  $t \in [0, T]$ ,*

$$\lim_{\varepsilon \rightarrow 0} |P_t^0(\Phi) - P_t^\varepsilon(\Phi)| = 0.$$

*Proof.* Let  $\varepsilon$  be sufficiently small so that lemma 3.3.35 holds. We prove by induction on  $n$ , the number of collisions in  $\Phi$ . Suppose  $n = 0$ . Then by definition 3.3.21, (3.3.37) and lemma 3.3.35,

$$\begin{aligned} |P_t^0(\Phi) - P_t^\varepsilon(\Phi)| &= \left| \exp \left( - \int_0^t L_s^0(\Phi) \, ds \right) - \exp \left( - \int_0^t L_s^\varepsilon(\Phi) \, ds \right) \right| f_0(x_0, v_0) \\ &\leq \int_0^t |L_s^0(\Phi) - L_s^\varepsilon(\Phi)| \, ds f_0(x_0, v_0) \\ &\leq 2CT(1 + |v(\tau)|) \varepsilon^{\alpha/6} f_0(x_0, v_0), \end{aligned}$$

as required. Now suppose the result holds true for almost all  $\Phi \in \mathcal{MT}$  with  $n = j$  for some  $j \geq 0$  and let  $\Psi \in \mathcal{MT}$  with  $n = j + 1$  be such that the result holds for  $\bar{\Psi}$  and,

$$\begin{aligned} \bar{g}(v')(1 + |v'|) &\leq M_\infty \text{ and} \\ |g_\tau(x(\tau), v') - g_\tau(x(\tau) + \varepsilon\nu, v')| &\leq M\varepsilon^\alpha. \end{aligned}$$

Indeed by (3.2.4) and (3.2.6) this only excludes a set of zero measure. Let  $\delta > 0$ . Then using (3.2.6) take  $\varepsilon$  sufficiently small so that,

$$|g_\tau(x(\tau), v') - g_\tau(x(\tau) + \varepsilon\nu, v')| \leq M\varepsilon^\alpha < \frac{\delta}{3(1 + P_\tau^0(\bar{\Psi}))(1 + |v(\tau^-)| + |v'|)}. \quad (3.3.74)$$

And using the inductive assumption take  $\varepsilon$  sufficiently small so that,

$$|P_\tau^0(\bar{\Psi}) - P_\tau^\varepsilon(\bar{\Psi})| < \frac{\delta}{3M_\infty(1 + |v(\tau^-)|)}. \quad (3.3.75)$$

Now by the inductive assumption for  $\varepsilon$  sufficiently small,

$$0 \leq P_\tau^\varepsilon(\bar{\Psi}) \leq |P_\tau^0(\bar{\Psi}) - P_\tau^\varepsilon(\bar{\Psi})| + P_\tau^0(\bar{\Psi}) \leq 1 + P_\tau^0(\bar{\Psi}).$$

So, as in the base case, take  $\varepsilon$  sufficiently small so that,

$$\begin{aligned} \left| \exp \left( - \int_{\tau}^t L_s^0(\Psi) \, ds \right) - \exp \left( - \int_{\tau}^t L_s^{\varepsilon}(\Psi) \, ds \right) \right| &\leq \int_{\tau}^t |L_s^0(\Psi) - L_s^{\varepsilon}(\Psi)| \, ds \\ &\leq \frac{\delta}{3(1 + P_{\tau}^{\varepsilon}(\bar{\Psi}))M_{\infty}(1 + |v(\tau^{-}) \cdot \nu|)}. \end{aligned} \quad (3.3.76)$$

Hence by (3.3.74), (3.3.75) and (3.3.76) and bounding the exponential term by 1, for  $\varepsilon$  sufficiently small,

$$\begin{aligned} |P_t^0(\Psi) - P_t^{\varepsilon}(\Psi)| &= \left| \exp \left( - \int_{\tau}^t L_s^0(\Phi) \, ds \right) P_{\tau}^0(\Psi) - \exp \left( - \int_{\tau}^t L_s^{\varepsilon}(\Phi) \, ds \right) P_{\tau}^{\varepsilon}(\Psi) \right| \\ &= \left| \exp \left( - \int_{\tau}^t L_s^0(\Phi) \, ds \right) P_{\tau}^0(\bar{\Psi})g_{\tau}(x(\tau), v')[(v(\tau^{-}) - v') \cdot \nu]_{+} \right. \\ &\quad \left. - \exp \left( - \int_{\tau}^t L_s^{\varepsilon}(\Phi) \, ds \right) P_{\tau}^{\varepsilon}(\bar{\Psi})g_{\tau}(x(\tau) + \varepsilon\nu, v')[(v(\tau^{-}) - v') \cdot \nu]_{+} \right| \\ &\leq P_{\tau}^0(\bar{\Psi})[(v(\tau^{-}) - v') \cdot \nu]_{+} |g_{\tau}(x(\tau), v') - g_{\tau}(x(\tau) + \varepsilon\nu, v')| \\ &\quad + g_{\tau}(x(\tau) + \varepsilon\nu, v')[(v(\tau^{-}) - v') \cdot \nu]_{+} |P_{\tau}^0(\bar{\Psi}) - P_{\tau}^{\varepsilon}(\bar{\Psi})| \\ &\quad + P_{\tau}^{\varepsilon}(\bar{\Psi})g_{\tau}(x(\tau) + \varepsilon\nu, v')[(v(\tau^{-}) - v') \cdot \nu]_{+} \\ &\quad \times \left| \exp \left( - \int_{\tau}^t L_s^0(\Phi) \, ds \right) - \exp \left( - \int_{\tau}^t L_s^{\varepsilon}(\Phi) \, ds \right) \right| \\ &< \delta. \end{aligned}$$

This completes the inductive step and so proves the result.  $\square$

We can now prove the remainder of theorem 3.3.1.

*Proof of Theorem 3.3.1.* Let  $\varepsilon \geq 0$ . By proposition 3.3.27 it remains only to prove that  $P_t^{\varepsilon}$  is a probability measure and that (3.3.8) holds. Positivity follows by the definition of  $P_t^{\varepsilon}$  in definition 3.3.21. By (3.3.7),

$$\int_{\mathcal{MT}} P_0^{\varepsilon}(\Phi) \, d\Phi = \int_{U \times \mathbb{R}^3} f_0(x, v) \, dx \, dv = 1.$$

Now let  $t > 0$ . By lemmas 3.3.33 and 3.3.34,  $\partial_t \int_{\mathcal{MT}} P_t^{\varepsilon}(\Phi) \, d\Phi$  exists and is equal to zero. Moreover by lemma 3.3.33  $\partial_t|_{t=0} \int_{\mathcal{MT}} P_t^{\varepsilon}(\Phi) \, d\Phi = 0$ . Hence,

$$\int_{\mathcal{MT}} P_t^{\varepsilon}(\Phi) \, d\Phi = 1.$$

It remains to prove (3.3.8). Since  $P_t^\varepsilon$  and  $P_t^0$  are probability measures on  $\mathcal{MT}$  and we have proven pointwise convergence in lemma 3.3.36 we apply Scheffé's theorem (see [7, Theorem 16.12]) which immediately gives the result.  $\square$

We finish this section by proving that  $f_t^\varepsilon$ , the evolution semigroup solution to the  $\varepsilon$  dependent linear Boltzmann equation, is a probability measure and that it converges in total variation to  $f_t^0$  as  $\varepsilon$  tends to zero.

**Proposition 3.3.37.** *For any  $t \in [0, T]$  and  $\varepsilon \geq 0$ ,  $f_t^\varepsilon$  is a probability measure on  $U \times \mathbb{R}^3$  and the trajectory  $V^\varepsilon(t, 0)f_0$  is honest (see [4, Remark 4.20]). Moreover  $f_t^\varepsilon$  converges to  $f_t^0$  in total variation as  $\varepsilon$  tends to zero uniformly for  $t \in [0, T]$ .*

*Proof.* Let  $t \in [0, T], \varepsilon \geq 0$ . Since  $f_0 \in L_1^+(U \times \mathbb{R}^3)$  we have by [4, proposition 2.2] for any  $j \geq 0$ ,  $V_j^\varepsilon(t, 0)f_0 \in L_1^+(U \times \mathbb{R}^3)$ , where  $V_j^\varepsilon$  are as in the proof of proposition 3.3.26. Since  $V^\varepsilon = \sum_{j=0}^\infty V_j^\varepsilon$  it follows that  $V^\varepsilon(t, 0)f_0 \in L_1^+(U \times \mathbb{R}^3)$ . Now by theorem 3.3.1 and (3.3.52),

$$\int_{U \times \mathbb{R}^3} f_t^\varepsilon(x, v) dx dv = \int_{\mathcal{MT}} P_t^\varepsilon(\Phi) d\Phi = 1,$$

so  $f_t^\varepsilon$  is a probability measure. Further this implies,

$$\int_{U \times \mathbb{R}^3} V^\varepsilon(t, 0)f_0(x, v) dx dv = \int_{U \times \mathbb{R}^3} f_0(x, v) dx dv.$$

Honesty of the trajectory of  $V^\varepsilon(t, 0)f_0$  follows from [4, section 4.3]. To prove convergence in total variation let  $t \in [0, T]$  and  $\Omega \subset U \times \mathbb{R}^3$ . Then by theorem 3.3.1,

$$\begin{aligned} \left| \int_{\Omega} f_t^0(x, v) - f_t^\varepsilon(x, v) dx dv \right| &= \left| \int_{S_t(\Omega)} P_t^0(\Phi) - P_t^\varepsilon(\Phi) d\Phi \right| \\ &\leq \int_{\mathcal{MT}} |P_t^0(\Phi) - P_t^\varepsilon(\Phi)| d\Phi \\ &\rightarrow 0, \end{aligned}$$

as required.  $\square$

### 3.4 The Empirical Distribution

We now describe the empirical distribution  $\hat{P}_t^\varepsilon$ . The main result of this section is theorem 3.4.17, where we show that  $\hat{P}_t^\varepsilon$  solves the empirical equation - at least for well controlled histories. The similarity of the empirical and idealised equations is then used



in section 3.5 to prove the convergence between  $P_t^\varepsilon$  and  $\hat{P}_t^\varepsilon$ , which is used to prove the required convergence of theorem 3.2.4.

### 3.4.1 Defining the Empirical Distribution

In this subsection we rigorously define the empirical distribution  $\hat{P}_t^\varepsilon$  on the set of collision histories  $\mathcal{MT}$  and prove its relation to  $\hat{f}_t^N$  the distribution of the tagged particle.

Recall that  $\hat{f}_t^N$  denotes the distribution of the tagged particle evolving via the Rayleigh gas dynamics amongst  $N$  background particles. In fact  $\hat{f}_t^N$  can be seen as the first marginal of the full  $N + 1$  particle distribution on  $(U \times \mathbb{R}^3)^{N+1}$ , which we denote by  $\hat{f}_t^{N,N+1}$ . Then for any  $(x, v) \in U \times \mathbb{R}^3$ ,

$$\hat{f}_t^N(x, v) = \int_{(U \times \mathbb{R}^3)^N} \hat{f}_t^{N,N+1}(x, v, x_1, v_1, \dots, x_N, v_N) dx_1 dv_1 \dots dx_N dv_N.$$

Notice that on the set  $D_N$  defined by,

$$D_N := \{(x, v, x_1, v_1, \dots, x_N, v_N) : |x - x_j| > \varepsilon \text{ for all } j = 1, \dots, N\},$$

(that is the subspace where all particles travel in free flow and there are no collisions) we have, with the convention that  $(x_0, v_0) = (x, v)$ ,

$$\partial_t \hat{f}_t^{N,N+1} + \sum_{i=0}^N v_i \cdot \nabla_{x_i} \hat{f}_t^{N,N+1} = 0.$$

Furthermore for any pre-collisional configuration of the  $N + 1$  particles denoted  $Z_{in}$  with corresponding post-collisional configuration  $Z_{out}$  we have,

$$\hat{f}_t^{N,N+1}(Z_{in}) = \hat{f}_t^{N,N+1}(Z_{out}).$$

Finally we note that since all the background particles are indistinguishable we have that  $\hat{f}_t^{N,N+1}$  is symmetric for all but the first two inputs  $(x, v)$ .

Before we can rigorously define the empirical distribution we first define a number of concepts and properties.

**Definition 3.4.1.** A collision occurring at time  $t$  between the tagged particle and background particle  $j$  is called a *grazing collision* if

$$(v(t^-) - v_j(t^-)) \cdot \nu = 0.$$

We refer to [20, Prop 4.1.1] for a proof that the set of initial positions that lead to a grazing collision is of zero measure.

**Definition 3.4.2.** For a collision history  $\Phi \in \mathcal{MT}$  let  $\bar{n}$  denote the number of distinct background particles involved in  $\Phi$ . Note that if each background particle in  $\Phi$  is distinct, that is there are no re-collisions, then  $\bar{n} = n$ . Otherwise  $\bar{n} < n$  and at least one of the background particles in  $\Phi$  has two collisions with the tagged particle.

**Definition 3.4.3.** For a collision history  $\Phi \in \mathcal{MT}$  and a given  $\varepsilon > 0$  define  $T^\varepsilon(\Phi) \in (U \times \mathbb{R}^3)^{\bar{n}+1}$  to be the initial positions and velocities of the tagged particle and each background particle involved in  $\Phi$ . The initial position of the tagged particle is given in  $\Phi$  and for each of the background particles one can work backwards given that it is known that it collides with the tagged particle at a given time with a given collision parameter.

We now define the concept of impossible collision histories, which are histories which we can write down but do not correspond with the physical dynamics of the system.

**Definition 3.4.4.** Let  $\varepsilon > 0$ . A collision history  $\Phi \in \mathcal{MT}$  with corresponding initial positions of  $\bar{n} + 1$  particles  $T^\varepsilon(\Phi)$  is called *impossible at diameter  $\varepsilon$*  if initiating the  $\bar{n} + 1$  particles involved in the history by the positions given in  $T^\varepsilon(\Phi)$  would not lead to  $\Phi$ .

This happens if one of the background particles, starting from its given position from  $T^\varepsilon(\Phi)$ , cannot reach the position where it collides with the tagged particle without previously interfering with the tagged particle's path. See figure 3-1.

Denote the set of all impossible collision histories at diameter  $\varepsilon$  by  $\mathcal{I}(\varepsilon) \subset \mathcal{MT}$ .

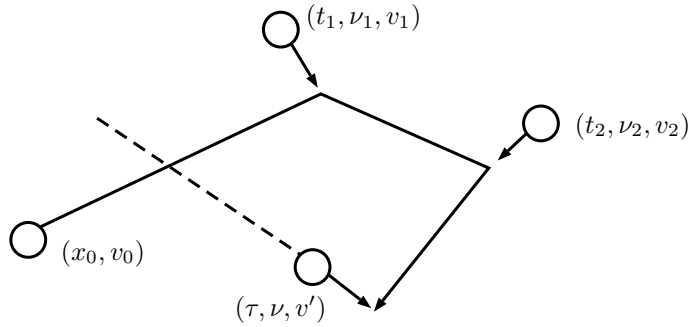


Figure 3-1: An example of an impossible history. We can trace all the particles back to their initial positions given by  $T^\varepsilon(\Phi)$ . However if we evolved the particles from this initial position the background particle corresponding to the final collision  $(\tau, \nu, v')$  would collide with the tagged particle where the dotted line intersects the tagged particle's path. Therefore this collision history is impossible and cannot occur.

**Definition 3.4.5.** Let  $N \in \mathbb{N}$ . Define  $\text{Prob}_N$  to be the probability measure on  $(U \times \mathbb{R}^3)^{N+1}$  induced by the random initial positions of the tagged particle and the  $N$  background particles.

Note that for any  $\Omega \subset U \times \mathbb{R}^3$  we have  $\text{Prob}((x_0, v_0) \in \Omega) = \int_{\Omega} f_0(x, v) dx dv$  and that for any  $(u_0, w_0), (u_1, w_1), \dots, (u_N, w_N) \in (U \times \mathbb{R}^3)^{N+1}$ ,

$$\begin{aligned} \text{Prob}_N((x_0, v_0), (x_1, v_1), \dots, (x_N, v_N) = (u_0, w_0), (u_1, w_1), \dots, (u_N, w_N)) \\ = f_0(u_0, w_0) \prod_{i=1}^N g_0(u_i, w_i). \end{aligned}$$

Since the only randomness in the Rayleigh gas particle dynamics we are considering is in the initial data we note that the probability of seeing any event defined by the particles can be traced back to the probability of seeing the appropriate initial data and so can be written in terms of  $\text{Prob}_N$ .

**Definition 3.4.6.** For a given history  $\Phi$ , a time  $t \in [0, T]$  and  $\varepsilon > 0$ , define the function  $\mathbb{1}_t^\varepsilon[\Phi] : U \times \mathbb{R}^3 \rightarrow \{0, 1\}$  by

$$\mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) := \begin{cases} 1 & \text{if for all } s \in (0, t), |x(s) - (\bar{x} + s\bar{v})| > \varepsilon, \\ 0 & \text{else.} \end{cases} \quad (3.4.1)$$

That is  $\mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v})$  is 1 if a background particle starting at the position  $(\bar{x}, \bar{v})$  avoids colliding with the tagged particle defined by the history  $\Phi$  up to the time  $t$  and zero otherwise.

We can now define the empirical distribution.

**Definition 3.4.7.** Let  $\Phi \in \mathcal{MT}$ . Define  $\hat{T}^\varepsilon(\Phi)$  to be the event that the tagged particle and  $\bar{n}$  background particles labelled  $i_1, \dots, i_{\bar{n}}$  have the initial positions  $T^\varepsilon(\Phi)$  - i.e. the initial positions that lead to the history  $\Phi$ .

Let  $S \subset \mathcal{MT}$  be open. Define

$$\begin{aligned} \hat{P}_t^\varepsilon(S) := \text{Prob}_N \left( \hat{T}^\varepsilon(\Phi) \text{ and } \mathbb{1}_t^\varepsilon[\Phi](x_i, v_i) = 1 \text{ for } i \in \{1, \dots, N\} \setminus \{i_1, \dots, i_{\bar{n}}\} \right. \\ \left. : \Phi \in S \setminus \mathcal{I}(\varepsilon) \right) \end{aligned}$$

That is for each  $\Phi \in S$ , excluding any  $\Phi$  that is in  $\mathcal{I}(\varepsilon)$  (since these histories are impossible and so have probability zero) we consider the probability that the tagged particle and the required number of background particles start at the required

positions and that all the other background particles have initial data such that they do not interfere with the tagged particle up to time  $t$ . This is exactly the initial conditions that lead to  $\Phi$  occurring.

We note that for  $S \subset \mathcal{MT}$  open the set of initial positions that lead to a history in  $S$  is also open since the resulting history continuously depends on the initial position of the particles, up to excluding the set of histories that include grazing collisions which result from a zero measure set of initial positions.

For any  $S \subset \mathcal{I}(\varepsilon) \subset \mathcal{MT}$ , we define  $\hat{P}_t^\varepsilon(S) := 0$ .

**Remark 3.4.8.** Indeed  $\hat{P}_t^\varepsilon$  is a probability measure on  $\mathcal{MT}$ . Positivity follows immediately. That  $\hat{P}_t^\varepsilon$  has unit mass follows from the fact that each initial state of the particles leads to a collision history and so considering all collision histories gives all possible initial positions and since  $\text{Prob}_N$  is a probability measure this has unit mass. Sigma additivity follows from the sigma additivity of  $\text{Prob}_N$ .

**Proposition 3.4.9.** *For any  $t \in [0, T]$  and any  $\Omega \subset U \times \mathbb{R}^3$  let  $S_t(\Omega) = \{\Phi \in \mathcal{MT} : (x(t), v(t)) \in \Omega\}$  as before. Then,*

$$\int_{\Omega} \hat{f}_t^N(x, v) \, dx \, dv = \hat{P}_t^\varepsilon(S_t(\Omega)).$$

*Proof.* As before for  $j \in \mathbb{N} \cup \{0\}$  define  $S_t^j(\Omega) = S_t(\Omega) \cap \mathcal{T}_j$  - i.e. histories with exactly  $j$  collisions such that the tagged particle is in  $\Omega$  at time  $t$ . Note that the  $S_t^j(\Omega)$  are disjoint.

Using the definition of  $\hat{P}_t^N$ , that  $\Phi \in S_t^j(\Omega)$  means that the tagged particle is in  $\Omega$  at time  $t$  and has experienced exactly  $j$  collisions and thus considering all  $\Phi \in S_t^j(\Omega)$  means we are considering all possible initial configurations such that the tagged particle

has exactly  $j$  collisions and is now in  $\Omega$  it follows that,

$$\begin{aligned}
\hat{P}_t^\varepsilon(S_t(\Omega)) &= \hat{P}_t^\varepsilon(\cup_{j=0}^\infty S_t^j(\Omega)) \\
&= \sum_{j=0}^\infty \hat{P}_t^\varepsilon(S_t^j(\Omega)) \\
&= \sum_{j=0}^\infty \text{Prob}_N \left( \hat{T}^\varepsilon(\Phi) \text{ and for } i \in \{1, \dots, N\} \setminus \{i_1, \dots, i_{\bar{n}}\} \mathbb{1}_t^\varepsilon[\Phi](x_i, v_i) = 1 \right. \\
&\quad \left. : \Phi \in S_t^j(\Omega) \setminus \mathcal{I}(\varepsilon) \right) \\
&= \sum_{j=0}^\infty \text{Prob}_N((x(t), v(t)) \in \Omega \text{ and the tagged particle has experienced} \\
&\quad \text{exactly } j \text{ collisions up to time } t) \\
&= \text{Prob}_N((x(t), v(t)) \in \Omega) \\
&= \int_{\Omega} \hat{f}_t^N(x, v) \, dx \, dv.
\end{aligned}$$

□

### 3.4.2 Good Histories

We now describe the set of ‘good’ collision histories on which we can calculate properties of  $\hat{P}_t^\varepsilon$  explicitly.

**Definition 3.4.10.** For a collision history  $\Phi \in \mathcal{MT}$  and time  $t \in [0, T]$  recall that we denote the position and velocity of the tagged particle by  $(x(t), v(t))$  and for  $j = 1, \dots, n$  the position and velocity of the background particle corresponding to the  $j$ -th collisions by  $(x_j(t), v_j(t))$ . Define  $\mathcal{V}(\Phi) \in [0, \infty)$  to be the maximum velocity involved in the history,

$$\mathcal{V}(\Phi) := \max_{t \in [0, T]} \left\{ |v(t)|, \max_{j=1, \dots, n(\Phi)} |v_j(t)| \right\}.$$

**Definition 3.4.11.** A history  $\Phi$  is called *re-collision free at diameter  $\varepsilon$*  if for all  $j = 1, \dots, n$  and for all  $t \in [0, T] \setminus \{t_j\}$  - where  $t_j$  denotes the time of collision between the tagged particle and background particle  $j$ ,

$$|x(t) - x_j(t)| > \varepsilon.$$

That is if the tagged particle collides with a background particle at time  $t_j$ , it has not collided with that background particle before in the history and up to time  $T$  it does

not come into contact with that particle again. Define

$$R(\varepsilon) := \{\Phi \in \mathcal{MT} : \Phi \text{ is re-collision free at diameter } \varepsilon\}.$$

**Definition 3.4.12.** A history  $\Phi \in \mathcal{MT}$  is called *non-grazing* if all collisions in  $\Phi$  are non-grazing, that is,

$$\min_{j=1, \dots, n(\Phi)} \nu_j \cdot (v(t_j^-) - v_j(t^-)) > 0.$$

**Definition 3.4.13.** A history  $\Phi \in \mathcal{MT}$  is called *free from initial overlap at diameter*  $\varepsilon > 0$  if initially the tagged particle is at least  $\varepsilon$  away from all the background particles. That is if for  $j = 1, \dots, N$

$$|x_0 - x_j| > \varepsilon.$$

we define,

$$S(\varepsilon) := \{\Phi \in \mathcal{MT} : \Phi \text{ is free from initial overlap at diameter } \varepsilon\}.$$

**Definition 3.4.14.** For any pair of decreasing functions  $V, M : (0, \infty) \rightarrow [0, \infty)$  such that  $\lim_{\varepsilon \rightarrow 0} V(\varepsilon) = \infty = \lim_{\varepsilon \rightarrow 0} M(\varepsilon)$  the set of *good histories of diameter*  $\varepsilon$  is defined by,

$$\mathcal{G}(\varepsilon) := \left\{ \Phi \in \mathcal{MT} : n(\Phi) \leq M(\varepsilon), \mathcal{V}(\Phi) < V(\varepsilon), \right. \\ \left. \Phi \in R(\varepsilon) \cap S(\varepsilon) \text{ and } \Phi \text{ is non-grazing} \right\}.$$

**Lemma 3.4.15.** *As  $\varepsilon$  decreases  $\mathcal{G}(\varepsilon)$  increases.*

*Proof.* The only non-trivial conditions are checking that  $S(\varepsilon)$  and  $R(\varepsilon)$  are increasing. To this aim suppose that  $\varepsilon' < \varepsilon$  and  $\Phi \in S(\varepsilon)$ . If  $n = 0$  then it follows from the definition that  $\Phi \in S(\varepsilon')$ . Else  $n \geq 1$ . For the background particles not involved in the history it is clear that reducing  $\varepsilon$  to  $\varepsilon'$  will not cause initial overlap. For  $1 \leq j \leq n$  the initial position of the background particle corresponding to collision  $j$  is  $x(t_j) - t_j v_j + \varepsilon \nu_j$ . Since  $\Phi \in S(\varepsilon)$ ,

$$|x_0 - (x(t_j) - t_j v_j + \varepsilon \nu_j)| > \varepsilon,$$

that is  $x_0 - (x(t_j) - t_j v_j) \notin B_\varepsilon(-\varepsilon \nu_j)$ . Hence  $x_0 - (x(t_j) - t_j v_j) \notin B_{\varepsilon'}(-\varepsilon' \nu_j)$  and so  $\Phi \in S(\varepsilon')$ .

Now suppose that  $\varepsilon' < \varepsilon$  and  $\Phi \notin R(\varepsilon')$ . Then in particular  $n \geq 1$  and there exists a  $1 \leq j \leq n$  and  $t > t_j$  such that, if we denote the velocity of the background particle

$j$  after its collision at time  $t_j$  by  $\bar{v}$ ,

$$x(t) - (x(t_j) + \varepsilon' \nu_j + (t - t_j) \bar{v}) \in \varepsilon' \mathbb{S}^2,$$

that is  $x(t) - (x(t_j) + (t - t_j) \bar{v}) \in \varepsilon' \nu_j + \varepsilon' \mathbb{S}^2$ . Hence since the left in side is continuous with respect to  $t$  it must be that there exists a  $t'$  such that,  $x(t') - (x(t_j) + (t' - t_j) \bar{v}) \in -\varepsilon \nu_j + \varepsilon \mathbb{S}^2$ , i.e.  $\Phi \notin R(\varepsilon)$ . Hence  $R(\varepsilon) \subset R(\varepsilon')$  and so  $\mathcal{G}(\varepsilon) \subset \mathcal{G}(\varepsilon')$ .  $\square$

We will later give restrictions on  $V$  and  $M$  in order to control bounds in order to prove required results.

**Lemma 3.4.16.** *Let  $\varepsilon > 0$  and  $\Phi \in \mathcal{G}(\varepsilon)$  then  $\hat{P}_t^\varepsilon$  is absolutely continuous with respect to the Lebesgue measure  $\lambda$  on a neighbourhood of  $\Phi$ .*

*Proof.* The proof follows in the same way as the proof of lemma 2.4.9 with the only difference being that instead of  $\int_{C_{h,j}} g_0(v) dx dv$ , since  $g_0$  now depends on  $x$  we have,  $\int_{C_{h,j}} g_0(x, v) dx dv$ . Because we are assuming  $g_0 \in L^1(U \times \mathbb{R}^3)$  the argument can be concluded in the same way.  $\square$

From now on we let  $\hat{P}_t^\varepsilon$  refer to the density of the probability measure on  $\mathcal{MT}$ .

### 3.4.3 The Empirical Equation

We now define the empirical equation, which we show  $\hat{P}_t^\varepsilon$  solves. First we define the operator  $\hat{Q}_t^\varepsilon$ , which is similar to the operator  $Q_t^\varepsilon$  in the idealised case, but includes the complexities of the particle evolution.

Recall the definition of  $\mathbb{1}_t^\varepsilon[\Phi]$  in (3.4.1). For  $\Phi \in \mathcal{MT}, t \geq 0$  and  $\varepsilon > 0$ , define the gain operator,

$$\hat{Q}_t^{\varepsilon,+}[\hat{P}_t](\Phi) := \begin{cases} \delta(t - \tau) \hat{P}_t(\bar{\Phi}) \frac{g_\tau(x(\tau) + \varepsilon \nu, v') [(v(\tau^-) - v') \cdot \nu]_+}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \end{cases}$$

and define the loss operator,

$$\hat{Q}_t^{\varepsilon,-}[\hat{P}_t](\Phi) := \hat{P}_t(\Phi) \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon \nu, \bar{v}) [(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}}.$$

For some  $\hat{C}(\varepsilon) > 0$  depending on  $t$  and  $\Phi$  of  $o(1)$  as  $\varepsilon$  tends to zero detailed later.

Finally define the operator  $\hat{\mathcal{Q}}_t^\varepsilon$  as follows,

$$\hat{\mathcal{Q}}_t^\varepsilon = \hat{\mathcal{Q}}_t^{\varepsilon,+} - \hat{\mathcal{Q}}_t^{\varepsilon,-}.$$

**Theorem 3.4.17.** *For  $\varepsilon$  sufficiently small and for all  $\Phi \in \mathcal{G}(\varepsilon)$ ,  $\hat{P}_t^\varepsilon$  solves the following*

$$\begin{cases} \partial_t \hat{P}_t^\varepsilon(\Phi) &= (1 - \gamma^\varepsilon(t)) \hat{\mathcal{Q}}_t^\varepsilon[\hat{P}_t^\varepsilon](\Phi) \\ \hat{P}_0^\varepsilon(\Phi) &= \zeta^\varepsilon(\Phi) f_0(x_0, v_0) \mathbb{1}_{n(\Phi)=0}. \end{cases} \quad (3.4.2)$$

The functions  $\gamma^\varepsilon$  and  $\zeta^\varepsilon$  are given by

$$\zeta^\varepsilon(\Phi) := \left( 1 - \int_{B_\varepsilon(x_0)} \int_{\mathbb{R}^3} g_0(\bar{x}, \bar{v}) \, d\bar{v} \, d\bar{x} \right)^N, \quad (3.4.3)$$

and,

$$\gamma^\varepsilon(t) := \begin{cases} 1 & \text{if } t < \tau \\ n(\bar{\Phi})\varepsilon^2 & \text{if } t = \tau \\ n(\Phi)\varepsilon^2 & \text{if } t > \tau. \end{cases}$$

We prove this theorem by breaking it into several lemmas proving the initial data, gain and loss term separately using the definition of  $\hat{P}_t^\varepsilon$ . Firstly, the initial condition requirement for  $\hat{P}_t^\varepsilon$ .

**Definition 3.4.18.** Let  $\omega_0 \in U \times \mathbb{R}^3$  be the random initial position and velocity of the tagged particle. For  $1 \leq j \leq N$  let  $\omega_j$  be the random initial position and velocity of the  $j$ th background particle. By our assumptions  $\omega_0$  has distribution  $f_0$  and each  $\omega_j$  has distribution  $g_0$ . Finally let  $\omega = (\omega_1, \dots, \omega_N)$ .

**Lemma 3.4.19.** *Under the assumptions and set up of theorem 3.4.17 we have,*

$$\hat{P}_0^\varepsilon(\Phi) = \zeta^\varepsilon(\Phi) f_0(x_0, v_0) \mathbb{1}_{n(\Phi)=0}.$$

*Proof.* If  $n(\Phi) > 0$ ,  $\hat{P}_0(\Phi) = 0$ , because the history involves collisions happening at some positive time and as such cannot have occurred at time 0.

Else  $n(\Phi) = 0$ , so  $\Phi$  contains only the tagged particle. The probability of finding the tagged particle at the given initial data  $(x_0, v_0)$  is  $f_0(x_0, v_0)$ . But this must be multiplied by a factor less than one because we rule out situations that give initial



overlap of the tagged particle with a background particle. Firstly we calculate,

$$\begin{aligned}\mathbb{P}(|x_0 - x_1| > \varepsilon) &= 1 - \mathbb{P}(|x_0 - x_1| < \varepsilon) = 1 - \int_{\mathbb{R}^3} \int_{|x_0 - x_1| < \varepsilon} g_0(x_1, \bar{v}) \, dx_1 \, d\bar{v} \\ &= 1 - \int_{B_\varepsilon(x_0)} \int_{\mathbb{R}^3} g_0(\bar{x}, \bar{v}) \, d\bar{v} \, d\bar{x}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{P}(|x_0 - x_j| > \varepsilon, \forall j = 1, \dots, N) &= \mathbb{P}(|x_0 - x_1| > \varepsilon)^N \\ &= \left(1 - \int_{B_\varepsilon(x_0)} \int_{\mathbb{R}^3} g_0(\bar{x}, \bar{v}) \, d\bar{v} \, d\bar{x}\right)^N = \zeta^\varepsilon(\Phi).\end{aligned}$$

□

**Lemma 3.4.20.** *Under the set up of Theorem 3.4.17 for  $n \geq 1$ ,*

$$\hat{P}_\tau^\varepsilon(\Phi) = (1 - \gamma^\varepsilon(\tau)) P_\tau^\varepsilon(\bar{\Phi}) \frac{g_\tau(x(\tau) + \varepsilon\nu, v')[(v(\tau^-) - v') \cdot \nu]_+}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}}.$$

*Proof.* We can follow the proof of lemma 2.4.19 by replacing  $g_0(v')$  with  $g_\tau(x(\tau) + \varepsilon\nu, v')$ . □

From now on we make the following assumptions on the functions  $V, M$  in the definition 3.4.14. Assume that for any  $0 < \varepsilon < 1$  we have,

$$\varepsilon V(\varepsilon)^3 \leq \frac{1}{8} \text{ and } M(\varepsilon) \leq \frac{1}{\sqrt{\varepsilon}}. \quad (3.4.4)$$

Before we can prove the loss term we require a number of lemmas that are used to justify that  $\hat{P}_t^\varepsilon$  is differentiable for  $t > \tau$  and has the required derivative.

**Definition 3.4.21.** Let  $\varepsilon > 0$  and  $\Phi \in \mathcal{G}(\varepsilon)$ . For  $h > 0$  define,

$$\begin{aligned}W_{t,h}^\varepsilon(\Phi) &:= \left\{ (\bar{x}, \bar{v}) \in U \times \mathbb{R}^3 : \exists (t', \nu') \in (t, t+h) \times \mathbb{S}^2 \right. \\ &\quad \left. \text{such that } x(t') + \varepsilon\nu' = \bar{x} + t'\bar{v} \text{ and } (v(t') - \bar{v}) \cdot \nu' > 0 \right\}.\end{aligned}$$

That is  $W_{t,h}^\varepsilon(\Phi)$  contains all possible initial points for a background particle to start such that, if it travels with constant velocity, it will collide the tagged particle at some time in  $(t, t+h)$ . Further define,

$$I_{t,h}^\varepsilon(\Phi) := \frac{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_{W_{t,h}^\varepsilon(\Phi)}(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon(\Phi)(\bar{x}, \bar{v}) d\bar{x} d\bar{v}}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon(\Phi)(\bar{x}, \bar{v}) d\bar{x} d\bar{v}}.$$

**Lemma 3.4.22.** For  $\varepsilon > 0$  sufficiently small,  $\Phi \in \mathcal{G}(\varepsilon)$  and  $t > \tau$ ,

$$\lim_{h \downarrow 0} \frac{1}{h} \hat{P}_t^\varepsilon(\#(\omega \cap W_{t,h}^\varepsilon(\Phi)) \geq 2 \mid \Phi) = 0. \quad (3.4.5)$$

and,

$$\lim_{h \downarrow 0} \frac{1}{h} \hat{P}_{t-h}^\varepsilon(\#(\omega \cap W_{t-h,h}^\varepsilon(\Phi)) \geq 2 \mid \Phi) = 0. \quad (3.4.6)$$

*Proof.* We first prove (3.4.5). Since  $\Phi$  is a good history, so in particular is re-collision free, and we are conditioning on  $\Phi$  occurring at time  $t$ , we know that for  $1 \leq j \leq n$ ,  $\omega_j \notin W_{t,h}^\varepsilon(\Phi)$  (since if this was not the case there would be a re-collision). Hence by the independence of the initial distribution of the background particles,

$$\begin{aligned} \hat{P}_t^\varepsilon(\#(\omega \cap W_{t,h}^\varepsilon(\Phi)) \geq 2 \mid \Phi) &= \hat{P}_t^\varepsilon\left(\bigcup_{n+1 \leq i < j \leq N} \{\omega_i \in W_{t,h}^\varepsilon(\Phi) \text{ and } \omega_j \in W_{t,h}^\varepsilon(\Phi)\} \mid \Phi\right) \\ &\leq \sum_{n+1 \leq i < j \leq N} \hat{P}_t^\varepsilon(\omega_i \in W_{t,h}^\varepsilon(\Phi) \text{ and } \omega_j \in W_{t,h}^\varepsilon(\Phi) \mid \Phi) \\ &\leq N^2 \hat{P}_t^\varepsilon(\omega_N \in W_{t,h}^\varepsilon(\Phi) \mid \Phi)^2. \end{aligned} \quad (3.4.7)$$

We note that,

$$\hat{P}_t^\varepsilon(\omega_N \in W_{t,h}^\varepsilon(\Phi) \mid \Phi) = I_{t,h}^\varepsilon(\Phi).$$

We now bound the numerator and denominator of  $I_{t,h}^\varepsilon(\Phi)$ . Firstly, for a fixed  $\bar{v}$ , the set of points  $\bar{x}$  such that  $(\bar{x}, \bar{v}) \in W_{h,t}^\varepsilon(\Phi)$  is a cylinder of radius  $\varepsilon$  and length  $\int_t^{t+h} |v(s) - \bar{v}| ds$ . Hence, since  $\Phi \in \mathcal{G}(\varepsilon)$ ,

$$\begin{aligned} \int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_{W_{t,h}^\varepsilon(\Phi)}(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon(\Phi)(\bar{x}, \bar{v}) d\bar{x} d\bar{v} &\leq \int_{U \times \mathbb{R}^3} \bar{g}(\bar{v}) \mathbb{1}_{W_{t,h}^\varepsilon(\Phi)}(\bar{x}, \bar{v}) d\bar{x} d\bar{v} \\ &= \pi \varepsilon^2 \int_{\mathbb{R}^3} \bar{g}(\bar{v}) \int_t^{t+h} |v(s) - \bar{v}| ds d\bar{v} \leq h \pi \varepsilon^2 \int_{\mathbb{R}^3} \bar{g}(\bar{v}) (V(\varepsilon) + |\bar{v}|) d\bar{v} \\ &\leq h \pi \varepsilon^2 M_g(V(\varepsilon) + 1). \end{aligned} \quad (3.4.8)$$

Now turning to the denominator, we note first that

$$\begin{aligned} \int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon(\Phi)(\bar{x}, \bar{v}) d\bar{x} d\bar{v} &= \int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) (1 - \mathbb{1}_{W_{0,t}^\varepsilon(\Phi)}(\bar{x}, \bar{v})) d\bar{x} d\bar{v} \\ &= 1 - \int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_{W_{0,t}^\varepsilon(\Phi)}(\bar{x}, \bar{v}) d\bar{x} d\bar{v}. \end{aligned}$$

By the same estimates as in the numerator, using  $t \in [0, T]$  we have,

$$\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_{W_{0,t}^\varepsilon(\Phi)}(\bar{x}, \bar{v}) d\bar{x} d\bar{v} \leq \varepsilon^2 T M_g(V(\varepsilon) + 1).$$

So by (3.4.4) we have that for  $\varepsilon$  sufficiently small this is less than  $1/2$ . Hence

$$\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon(\Phi)(\bar{x}, \bar{v}) d\bar{x} d\bar{v} \geq 1/2.$$

Combining this and (3.4.8) we have that for  $\varepsilon$  sufficiently small,

$$I_{t,h}^\varepsilon(\Phi) \leq 2h\pi\varepsilon^2 M_g(V(\varepsilon) + 1). \quad (3.4.9)$$

Hence substituting this into (3.4.7), and using that in the Boltzmann-Grad scaling  $N\varepsilon^2 = 1$ ,

$$\begin{aligned} \hat{P}_t^\varepsilon(\#(\omega \cap W_{t,h}^\varepsilon(\Phi)) \geq 2 \mid \Phi) &\leq N^2 \varepsilon^4 h^2 \pi^2 M_g^2(V(\varepsilon) + 1)^2 \\ &= h^2 \pi^2 M_g^2(V(\varepsilon) + 1)^2. \end{aligned}$$

Diving by  $h$  and taking the limit  $h \downarrow 0$  gives (3.4.5). For (3.4.6) we use the same argument to see that,

$$\hat{P}_{t-h}^\varepsilon(\#(\omega \cap W_{t-h,h}^\varepsilon(\Phi)) \geq 2 \mid \Phi) \leq N^2 \hat{P}_{t-h}^\varepsilon(\omega_N \in W_{t-h,h}^\varepsilon(\Phi) \mid \Phi)^2 = N^2 I_{t-h,h}^\varepsilon(\Phi)^2.$$

We can now employ a similar approach to show that for  $\varepsilon$  sufficiently small, after diving by  $h$ , this converges to zero as  $h \downarrow 0$ .  $\square$

**Definition 3.4.23.** For  $\Phi \in \mathcal{MT}$ ,  $t > \tau$ ,  $h > 0$  and  $\varepsilon > 0$  define

$$B_{t,h}^\varepsilon(\Phi) := \{(\bar{x}, \bar{v}) \in U \times \mathbb{R}^3 : \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) = 0 \text{ and } \mathbb{1}_{W_{t,h}^\varepsilon(\Phi)}(\bar{x}, \bar{v}) = 1\}$$

That is,  $B_{t,h}^\varepsilon(\Phi)$  is the set of all initial positions such that, if a background particles starts at  $(\bar{x}, \bar{v})$  and travels with constant velocity (even if it meets the tagged particle) it collides with the tagged particle once in  $(0, t)$  and again in  $(t, t+h)$ .

**Lemma 3.4.24.** For  $\varepsilon$  sufficiently small,  $\Phi \in \mathcal{G}(\varepsilon)$  and  $t > \tau$  there exists a  $\hat{C}(\varepsilon) > 0$  depending on  $t$  and  $\Phi$  with  $\hat{C}(\varepsilon) = o(1)$  as  $\varepsilon$  tends to zero, such that

$$\int_{B_{t,h}^\varepsilon(\Phi)} g_0(\bar{x}, \bar{v}) d\bar{x} d\bar{v} = \int_{B_{t-h,h}^\varepsilon(\Phi)} g_0(\bar{x}, \bar{v}) d\bar{x} d\bar{v} = h\varepsilon^2 \hat{C}(\varepsilon).$$

*Proof.* Using that for any  $\Omega \subset U \times \mathbb{R}^3$  measurable,

$$\int_{\Omega} g_0(\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v} \leq \int_{\Omega} \bar{g}(\bar{v}) \, d\bar{x} \, d\bar{v},$$

we can repeat the proof of lemma 2.4.14.  $\square$

**Lemma 3.4.25.** *For  $\varepsilon > 0$  sufficiently small,  $\Phi \in \mathcal{G}(\varepsilon)$  and  $t > \tau$ ,*

$$\begin{aligned} & \lim_{h \downarrow 0} \frac{1}{h} \hat{P}_t^\varepsilon(\#(\omega \cap W_{t,h}^\varepsilon(\Phi)) > 0 \mid \Phi) \\ &= (1 - \gamma^\varepsilon(t)) \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon\nu, \bar{v}) [(v(\tau) - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}}. \end{aligned} \quad (3.4.10)$$

and,

$$\begin{aligned} & \lim_{h \downarrow 0} \frac{1}{h} \hat{P}_{t-h}^\varepsilon(\#(\omega \cap W_{t-h,h}^\varepsilon(\Phi)) > 0 \mid \Phi) \\ &= (1 - \gamma^\varepsilon(t)) \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon\nu, \bar{v}) [(v(\tau) - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}}. \end{aligned} \quad (3.4.11)$$

*Proof.* We first show (3.4.10). By (3.4.5) we only need calculate,

$$\lim_{h \downarrow 0} \frac{1}{h} \hat{P}_t^\varepsilon(\#(\omega \cap W_{t,h}^\varepsilon(\Phi)) = 1 \mid \Phi).$$

Now by a similar argument to the proof of lemma 3.4.22 we have,

$$\begin{aligned} & \hat{P}_t^\varepsilon(\#(\omega \cap W_{t,h}^\varepsilon(\Phi)) = 1 \mid \Phi) \\ &= \sum_{i=n+1}^N \hat{P}_t^\varepsilon(\omega_i \in W_{t,h}^\varepsilon(\Phi) \text{ and for } n+1 \leq j \leq N, j \neq i, \omega_j \notin W_{t,h}^\varepsilon(\Phi) \mid \Phi) \\ &= (N-n) \hat{P}_t^\varepsilon(\omega_N \in W_{t,h}^\varepsilon(\Phi) \mid \Phi) \hat{P}_t^\varepsilon(\omega_{N-1} \notin W_{t,h}^\varepsilon(\Phi) \mid \Phi)^{N-n-1} \\ &= (N-n) I_{t,h}^\varepsilon(\Phi) (1 - I_{t,h}^\varepsilon(\Phi))^{N-n-1} \\ &= (N-n) \sum_{j=0}^{N-n-1} (-1)^j \binom{N-n-1}{j} I_{t,h}^\varepsilon(\Phi)^{j+1}. \end{aligned}$$

By (3.4.9) it follows that,

$$\begin{aligned}
\lim_{h \downarrow 0} \frac{1}{h} \hat{P}_t^\varepsilon(\#(\omega \cap W_{t,h}^\varepsilon(\Phi))) &= 1 \mid \Phi \\
&= \lim_{h \downarrow 0} \frac{1}{h} (N-n) \sum_{j=0}^{N-n-1} (-1)^j \binom{N-n-1}{j} I_{t,h}^\varepsilon(\Phi)^{j+1} \\
&= \lim_{h \downarrow 0} \frac{1}{h} (N-n) I_{t,h}^\varepsilon(\Phi).
\end{aligned} \tag{3.4.12}$$

We compute this limit by noting that,

$$\begin{aligned}
I_{t,h}^\varepsilon(\Phi) &= \frac{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_{W_{t,h}^\varepsilon(\Phi)}(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon(\Phi)(\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon(\Phi)(\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} \\
&= \frac{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_{W_{t,h}^\varepsilon(\Phi)}(\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon(\Phi)(\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} - \frac{\int_{B_{t,h}^\varepsilon(\Phi)} g_0(\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon(\Phi)(\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}}.
\end{aligned}$$

For the first term we note that since  $t > \tau$ , for any  $\bar{v} \in \mathbb{R}^3$

$$\begin{aligned}
\int_U \mathbb{1}_{W_{t,h}^\varepsilon(\Phi)}(\bar{x}, \bar{v}) \, d\bar{x} &= \pi \varepsilon^2 \int_t^{t+h} |v(s) - \bar{v}| \, ds \\
&= \pi \varepsilon^2 h |v(\tau) - \bar{v}|.
\end{aligned}$$

and for  $\bar{v} \in \mathbb{R}^3$  and  $t > \tau$ ,

$$\{(\bar{x}, \bar{v}) \in W_{t,0}^\varepsilon(\Phi)\} = \{(x(t) + \varepsilon \nu - t \bar{v}, \bar{v}) : \nu \in \mathbb{S}^2 \text{ and } (v(\tau) - \bar{v}) \cdot \nu > 0\}.$$

Hence for almost all  $\Phi \in \mathcal{G}(\varepsilon)$ ,

$$\begin{aligned}
\lim_{h \downarrow 0} \frac{1}{h} (N-n) \int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_{W_{t,h}^\varepsilon(\Phi)}(\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v} \\
&= \lim_{h \downarrow 0} \frac{1}{h} (N-n) \int_{W_{t,h}^\varepsilon(\Phi)} g_0(\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v} \\
&= (1 - \gamma^\varepsilon(t)) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon \nu, \bar{v}) [(v(\tau) - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu.
\end{aligned} \tag{3.4.13}$$

For the second term we have by lemma 3.4.24,

$$\lim_{h \downarrow 0} \frac{1}{h} (N-n) \int_{B_{t,h}^\varepsilon(\Phi)} g_0(\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v} = (N-n) \varepsilon^2 \hat{C}(\varepsilon) = (1 - \gamma^\varepsilon(t)) \hat{C}(\varepsilon). \tag{3.4.14}$$

Subtracting (3.4.14) from (3.4.13) and then dividing by  $\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon(\Phi)(\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}$

and substituting into (3.4.12) gives,

$$\begin{aligned} \lim_{h \downarrow 0} \frac{1}{h} \hat{P}_t^\varepsilon(\#(\omega \cap W_{t,h}^\varepsilon(\Phi))) \\ = (1 - \gamma^\varepsilon(t)) \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon \nu, \bar{v}) [(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}}, \end{aligned}$$

as required. For (3.4.11) we use (3.4.6) and take  $h > 0$  sufficiently small so that  $t-h > \tau$  and repeat the same argument. □

**Lemma 3.4.26.** *For  $\varepsilon > 0$  sufficiently small and  $\Phi \in \mathcal{G}(\varepsilon)$ ,  $\hat{P}^\varepsilon(\Phi) : (\tau, T] \rightarrow [0, \infty)$  is continuous with respect to  $t$ .*

*Proof.* Let  $t \in (\tau, T]$ . Then for  $h > 0$ ,

$$\hat{P}_{t+h}^\varepsilon(\Phi) = (1 - \hat{P}_t^\varepsilon(\#(\omega \cap W_{t,h}^\varepsilon(\Phi)) > 0 \mid \Phi)) \hat{P}_t^\varepsilon(\Phi). \quad (3.4.15)$$

Hence by (3.4.10),

$$\begin{aligned} |\hat{P}_{t+h}^\varepsilon(\Phi) - \hat{P}_t^\varepsilon(\Phi)| &= \hat{P}_t^\varepsilon(\#(\omega \cap W_{t,h}^\varepsilon(\Phi)) > 0 \mid \Phi) \hat{P}_t^\varepsilon(\Phi) \\ &\rightarrow 0 \text{ as } h \rightarrow 0. \end{aligned}$$

Now let  $h > 0$  be sufficiently small so that  $t-h > \tau$ . Then,

$$\hat{P}_t^\varepsilon(\Phi) = (1 - \hat{P}_{t-h}^\varepsilon(\#(\omega \cap W_{t-h,h}^\varepsilon(\Phi)) > 0 \mid \Phi)) \hat{P}_{t-h}^\varepsilon(\Phi). \quad (3.4.16)$$

Further by the properties of the particle dynamics we have, since there is a non-negative probability that the tagged particle experiences a collision in the time  $(\tau, t-h]$

$$\hat{P}_{t-h}^\varepsilon(\Phi) \leq \hat{P}_\tau^\varepsilon(\Phi).$$

Hence by (3.4.11),

$$\begin{aligned} |\hat{P}_t^\varepsilon(\Phi) - \hat{P}_{t-h}^\varepsilon(\Phi)| &= \hat{P}_{t-h}^\varepsilon(\#(\omega \cap W_{t-h,h}^\varepsilon(\Phi)) > 0 \mid \Phi) \hat{P}_{t-h}^\varepsilon(\Phi) \\ &\leq \hat{P}_{t-h}^\varepsilon(\#(\omega \cap W_{t-h,h}^\varepsilon(\Phi)) > 0 \mid \Phi) \hat{P}_\tau^\varepsilon(\Phi) \\ &\rightarrow 0 \text{ as } h \rightarrow 0. \end{aligned}$$

□

We can now prove that the loss term of (3.4.2) holds.

**Lemma 3.4.27.** For  $\varepsilon > 0$  sufficiently small and  $\Phi \in \mathcal{G}(\varepsilon)$ ,  $\hat{P}^\varepsilon(\Phi) : (\tau, T] \rightarrow [0, \infty)$  is differentiable and

$$\partial_t \hat{P}_t^\varepsilon(\Phi) = (1 - \gamma^\varepsilon(t)) \hat{Q}_t^\varepsilon[\hat{P}_t^\varepsilon](\Phi).$$

*Proof.* By (3.4.15) and (3.4.10),

$$\begin{aligned} \lim_{h \downarrow 0} \frac{1}{h} \left( \hat{P}_{t+h}^\varepsilon(\Phi) - \hat{P}_t^\varepsilon(\Phi) \right) &= \lim_{h \downarrow 0} \frac{1}{h} \left( \hat{P}_t^\varepsilon(\#(\omega \cap W_{t,h}^\varepsilon(\Phi)) > 0 \mid \Phi) \right) \hat{P}_t^\varepsilon(\Phi) \\ &= (1 - \gamma^\varepsilon(t)) \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon\nu, \bar{v}) [(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} \hat{P}_t^\varepsilon(\Phi) \\ &= (1 - \gamma^\varepsilon(t)) \hat{Q}_t^\varepsilon[\hat{P}_t^\varepsilon](\Phi). \end{aligned} \quad (3.4.17)$$

Further by (3.4.16), (3.4.11) and lemma 3.4.26 we have,

$$\begin{aligned} \lim_{h \downarrow 0} \frac{1}{h} \left( \hat{P}_t^\varepsilon(\Phi) - \hat{P}_{t-h}^\varepsilon(\Phi) \right) &= \lim_{h \downarrow 0} \frac{1}{h} \left( \hat{P}_{t-h}^\varepsilon(\#(\omega \cap W_{t-h,h}^\varepsilon(\Phi)) > 0 \mid \Phi) \right) \hat{P}_{t-h}^\varepsilon(\Phi) \\ &= (1 - \gamma^\varepsilon(t)) \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon\nu, \bar{v}) [(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} \hat{P}_t^\varepsilon(\Phi) \\ &= (1 - \gamma^\varepsilon(t)) \hat{Q}_t^\varepsilon[\hat{P}_t^\varepsilon](\Phi). \end{aligned} \quad (3.4.18)$$

Combining (3.4.17) and (3.4.18) proves the result.  $\square$

*Proof of theorem 3.4.17.* The result now follows by lemmas 3.4.19, 3.4.20 and 3.4.27.  $\square$

### 3.5 Convergence

We have proved that there exists a solution  $P_t^\varepsilon$  to the idealised equation in theorem 3.3.1 and we have shown in theorem 3.4.17 that the empirical distribution  $\hat{P}_t^\varepsilon$  solves the empirical equation, at least for good histories. Similarly to chapter 2 and [38] we now prove the convergence between  $P_t^0$  and  $\hat{P}_t^\varepsilon$ , which will enable the proof of theorem 3.2.4. This section closely follows section 2.5 the main difference being that because the initial distribution  $g_0$  is now spatially inhomogeneous we take an extra step by comparing  $P_t^\varepsilon$  and  $\hat{P}_t^\varepsilon$ . The change in the particle dynamics from chapter 2, where we now assume that the background particles change velocity when they collide with the tagged particle, makes only a minor difference to the proof.

### 3.5.1 Comparing $P_t^\varepsilon$ and $\hat{P}_t^\varepsilon$

We now introduce new notation. Recall the definition of  $\mathbb{1}_t^\varepsilon[\Phi]$  and  $\zeta^\varepsilon(\Phi)$  in (3.4.1) and (3.4.3) respectively. For  $\varepsilon > 0$ ,  $t \in [0, T]$  and  $\Phi \in \mathcal{G}(\varepsilon)$ ,

$$\begin{aligned}\eta_t^\varepsilon(\Phi) &:= \int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v})(1 - \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v})) \, d\bar{x} \, d\bar{v}, \\ R_t^\varepsilon(\Phi) &:= \zeta^\varepsilon(\Phi) P_t^\varepsilon(\Phi), \\ C^\varepsilon(\Phi) &:= 2 \sup_{t \in [0, T]} \left\{ \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon \nu, \bar{v}) [(v(t) - \bar{v}) \cdot \nu]_+ \, d\bar{v} \, d\nu \right\}, \\ \rho_t^{\varepsilon, 0}(\Phi) &:= \eta_t^\varepsilon(\Phi) C^\varepsilon(\Phi) t.\end{aligned}\tag{3.5.1}$$

Further for  $k \geq 1$  define, 1

$$\rho_t^{\varepsilon, k}(\Phi) := (1 - \varepsilon) \rho_t^{\varepsilon, k-1}(\Phi) + \rho_t^{\varepsilon, 0}(\Phi) + \varepsilon.\tag{3.5.2}$$

As in section 2.5 this recursive formula is used since we employ an inductive argument in the following proposition. Note that this implies that for  $k \geq 1$ ,

$$\rho_t^{\varepsilon, k}(\Phi) = (1 - \varepsilon)^k \rho_t^{\varepsilon, 0}(\Phi) + (\rho_t^{\varepsilon, 0}(\Phi) + \varepsilon) \sum_{j=1}^k (1 - \varepsilon)^{k-j}.\tag{3.5.3}$$

Finally define,

$$\hat{\rho}_t^\varepsilon(\Phi) := \rho_t^{\varepsilon, n(\Phi)}(\Phi).$$

**Proposition 3.5.1.** *For  $\varepsilon > 0$  sufficiently small, any  $t \in [0, T]$  and almost all  $\Phi \in \mathcal{G}(\varepsilon)$ ,*

$$\hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) \geq -\hat{\rho}_t^\varepsilon(\Phi) R_t^\varepsilon(\Phi).$$

To prove this proposition we use the following lemmas.

**Lemma 3.5.2.** *Let  $L_t^\varepsilon(\Phi)$  be given as in (3.3.5). For  $\Phi \in \mathcal{G}(\varepsilon)$  and  $t \geq \tau$ ,*

$$\begin{aligned}\hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) &\geq \exp\left(-\int_\tau^t (1 + 2\eta_s^\varepsilon(\Phi)) L_s^\varepsilon(\Phi) \, ds\right) (\hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi)) \\ &\quad - 2\eta_t^\varepsilon(\Phi) R_t^\varepsilon(\Phi) \int_\tau^t \exp\left(-2\eta_s^\varepsilon(\Phi) \int_s^t L_\sigma^\varepsilon(\Phi) \, d\sigma\right) L_s^\varepsilon(\Phi) \, ds.\end{aligned}$$

*Proof.* For  $t = \tau$  it is clear the result holds. Let  $t > \tau$ . By theorem 3.3.1 and theorem 3.4.17 we have that,

$$\partial_t \left( \hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) \right) = -(1 - \gamma^\varepsilon(t)) \hat{L}_t^\varepsilon(\Phi) \hat{P}_t^\varepsilon(\Phi) + L_t^\varepsilon(\Phi) R_t^\varepsilon(\Phi),\tag{3.5.4}$$



where,

$$\hat{L}_t^\varepsilon(\Phi) := \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon\nu, \bar{v})[(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}}.$$

Further by (3.2.2), (3.2.4) and (3.4.1),

$$\begin{aligned} \int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v})(1 - \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v})) d\bar{x} d\bar{v} &\leq \int_{U \times \mathbb{R}^3} \bar{g}(\bar{v})(1 - \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v})) d\bar{x} d\bar{v} \\ &\leq \int_{\mathbb{R}^3} \bar{g}(\bar{v}) \int_U (1 - \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v})) d\bar{x} d\bar{v} \leq \int_{\mathbb{R}^3} \bar{g}(\bar{v}) \left( \pi\varepsilon^2 \int_0^t |v(s) - \bar{v}| ds \right) d\bar{v} \\ &\leq \pi\varepsilon^2 \int_{\mathbb{R}^3} \bar{g}(\bar{v}) \left( \int_0^t V(\varepsilon) + |\bar{v}| ds \right) d\bar{v} \leq \pi\varepsilon^2 T \int_{\mathbb{R}^3} \bar{g}(\bar{v}) (V(\varepsilon) + |\bar{v}|) d\bar{v} \\ &\leq \pi\varepsilon^2 T M_g(V(\varepsilon) + 1). \end{aligned} \quad (3.5.5)$$

By (3.4.4) for  $\varepsilon$  sufficiently small we can make this less than  $1/2$ . Now using the fact that for  $0 \leq z \leq 1/2$  it follows  $1/(1-z) \leq 1+2z$ , we have for  $\varepsilon$  sufficiently small,

$$\begin{aligned} \frac{1}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} &= \frac{1}{1 - \int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v})(1 - \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v})) d\bar{x} d\bar{v}} \\ &\leq 1 + 2 \left( \int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v})(1 - \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v})) d\bar{x} d\bar{v} \right) \\ &= 1 + 2\eta_t^\varepsilon(\Phi). \end{aligned}$$

It follows that,

$$\begin{aligned} (1 - \gamma^\varepsilon(t)) &\left( \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon\nu, \bar{v})[(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu - \hat{C}(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} \right) \\ &\leq \frac{\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon\nu, \bar{v})[(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} \\ &\leq (1 + 2\eta_t^\varepsilon(\Phi)) \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon\nu, \bar{v})[(v(\tau) - \bar{v}) \cdot \nu]_+ d\bar{v} d\nu. \end{aligned}$$

This implies

$$-(1 - \gamma^\varepsilon(t)) \hat{L}_t^\varepsilon(\Phi) \geq -(1 + 2\eta_t^\varepsilon(\Phi)) L_t^\varepsilon(\Phi).$$

Returning to (3.5.4) we now see,

$$\begin{aligned} \partial_t \left( \hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) \right) &\geq -(1 + 2\eta_t^\varepsilon(\Phi)) L_t^\varepsilon(\Phi) \hat{P}_t^\varepsilon(\Phi) + L_t^\varepsilon(\Phi) R_t^\varepsilon(\Phi) \\ &= -(1 + 2\eta_t^\varepsilon(\Phi)) L_t^\varepsilon(\Phi) \left( \hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) \right) - 2\eta_t^\varepsilon(\Phi) L_t^\varepsilon(\Phi) R_t^\varepsilon(\Phi). \end{aligned}$$

For fixed  $\Phi$  this is a  $1d$  differential equation in  $t$  and so by the variation of constants

formula it follows that,

$$\begin{aligned}\hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) &\geq \exp\left(-\int_\tau^t (1 + 2\eta_s^\varepsilon(\Phi))L_s^\varepsilon(\Phi) \, ds\right) (\hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi)) \\ &\quad - \int_\tau^t \exp\left(-\int_s^t (1 + 2\eta_\sigma^\varepsilon(\Phi))L_\sigma^\varepsilon(\Phi) \, d\sigma\right) 2\eta_s^\varepsilon(\Phi)L_s^\varepsilon(\Phi)R_s^\varepsilon(\Phi) \, ds.\end{aligned}$$

Now from (3.4.1) we see that  $\mathbb{1}_t^\varepsilon[\Phi]$  is non-increasing in  $t$  and therefore  $\eta_t^\varepsilon(\Phi)$  is non-decreasing in  $t$ . Since  $L_t^\varepsilon(\Phi)$  is non-negative it follows that,

$$\begin{aligned}\hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) &\geq \exp\left(-\int_\tau^t (1 + 2\eta_s^\varepsilon(\Phi))L_s^\varepsilon(\Phi) \, ds\right) (\hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi)) \\ &\quad - 2\eta_t^\varepsilon(\Phi) \int_\tau^t \exp\left(-\int_s^t (1 + 2\eta_\sigma^\varepsilon(\Phi))L_\sigma^\varepsilon(\Phi) \, d\sigma\right) L_s^\varepsilon(\Phi)R_s^\varepsilon(\Phi) \, ds \\ &\geq \exp\left(-\int_\tau^t (1 + 2\eta_s^\varepsilon(\Phi))L_s^\varepsilon(\Phi) \, ds\right) (\hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi)) \\ &\quad - 2\eta_t^\varepsilon(\Phi) \int_\tau^t \exp\left(-(1 + 2\eta_s^\varepsilon(\Phi)) \int_s^t L_\sigma^\varepsilon(\Phi) \, d\sigma\right) L_s^\varepsilon(\Phi)R_s^\varepsilon(\Phi) \, ds.\end{aligned}\tag{3.5.6}$$

By definition 3.3.21 we have for  $\tau \leq s \leq t$ ,

$$R_t^\varepsilon(\Phi) = \exp\left(-\int_s^t L_\sigma^\varepsilon(\Phi) \, d\sigma\right) R_s^\varepsilon(\Phi),$$

implying for  $\tau \leq s \leq t$ ,

$$R_s^\varepsilon(\Phi) = \exp\left(\int_s^t L_\sigma^\varepsilon(\Phi) \, d\sigma\right) R_t^\varepsilon(\Phi).\tag{3.5.7}$$

Substituting this into (3.5.6) we have,

$$\begin{aligned}\hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) &\geq \exp\left(-\int_\tau^t (1 + 2\eta_s^\varepsilon(\Phi))L_s^\varepsilon(\Phi) \, ds\right) (\hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi)) \\ &\quad - 2\eta_t^\varepsilon(\Phi) \int_\tau^t \exp\left(-(1 + 2\eta_s^\varepsilon(\Phi)) \int_s^t L_\sigma^\varepsilon(\Phi) \, d\sigma\right) L_s^\varepsilon(\Phi) \\ &\quad \times \exp\left(\int_s^t L_\sigma^\varepsilon(\Phi) \, d\sigma\right) R_t^\varepsilon(\Phi) \, ds \\ &= \exp\left(-\int_\tau^t (1 + 2\eta_s^\varepsilon(\Phi))L_s^\varepsilon(\Phi) \, ds\right) (\hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi)) \\ &\quad - 2\eta_t^\varepsilon(\Phi) R_t^\varepsilon(\Phi) \int_\tau^t \exp\left(-2\eta_s^\varepsilon(\Phi) \int_s^t L_\sigma^\varepsilon(\Phi) \, d\sigma\right) L_s^\varepsilon(\Phi) \, ds,\end{aligned}$$

as required. □

**Lemma 3.5.3.**

1. For  $\Phi \in \mathcal{G}(\varepsilon)$  and  $t \geq \tau$ ,

$$2\eta_t^\varepsilon(\Phi) \int_\tau^t \exp\left(-(1 + 2\eta_s^\varepsilon(\Phi)) \int_s^t L_\sigma^\varepsilon(\Phi) d\sigma\right) L_s^\varepsilon(\Phi) ds \leq \rho_t^{\varepsilon,0}(\Phi).$$

2. For  $\varepsilon$  sufficiently small and almost all  $\Phi \in \mathcal{G}(\varepsilon)$  and any  $t \in [0, T]$ ,

$$1 - \frac{1 - \gamma^\varepsilon(t)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} \leq \varepsilon.$$

*Proof.* Let  $\Phi \in \mathcal{G}(\varepsilon)$  and  $t \geq \tau$ . To prove (1) note that we need to prove,

$$2 \int_\tau^t \exp\left(-(1 + 2\eta_s^\varepsilon(\Phi)) \int_s^t L_\sigma^\varepsilon(\Phi) d\sigma\right) L_s^\varepsilon(\Phi) ds \leq C^\varepsilon(\Phi)t.$$

Firstly by definition for any  $s \geq \tau$ ,  $L_s^\varepsilon(\Phi) \leq C^\varepsilon(\Phi)/2$ . Secondly since  $L_\sigma^\varepsilon \geq 0$ ,

$$\exp\left(-(1 + 2\eta_s^\varepsilon(\Phi)) \int_s^t L_\sigma^\varepsilon(\Phi) d\sigma\right) \leq 1.$$

Hence,

$$\begin{aligned} 2 \int_\tau^t \exp\left(-(1 + 2\eta_s^\varepsilon(\Phi)) \int_s^t L_\sigma^\varepsilon(\Phi) d\sigma\right) L_s^\varepsilon(\Phi) ds &\leq C^\varepsilon(\Phi) \int_\tau^t ds \\ &= C^\varepsilon(\Phi)(t - \tau) \leq C^\varepsilon(\Phi)t. \end{aligned}$$

We now prove (2). Repeating the argument of (3.5.5) we have,

$$\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) (1 - \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v})) d\bar{x} d\bar{v} \leq \pi\varepsilon^2 T M_g(V(\varepsilon) + 1),$$

which converges to zero as  $\varepsilon$  converges to zero by (3.4.4). Hence,

$$\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v} = 1 - \int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) (1 - \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v})) d\bar{x} d\bar{v},$$

converges to one as  $\varepsilon$  converges to zero. Now,

$$\begin{aligned}
& \frac{1}{\varepsilon} \left( 1 - \frac{1 - \gamma^\varepsilon(t)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} \right) \\
&= \frac{1}{\varepsilon} \left( \frac{1 - \int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) (1 - \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v})) \, d\bar{x} \, d\bar{v}}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} - \frac{1 - \gamma^\varepsilon(t)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} \right) \\
&\leq \frac{1}{\varepsilon} \left( \frac{\gamma^\varepsilon(t)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} \right) \\
&\leq \frac{\varepsilon n(\Phi)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} \leq \frac{\varepsilon M(\varepsilon)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}}.
\end{aligned}$$

By (3.4.4) the numerator converges to zero as  $\varepsilon$  converges to zero and the denominator converges to one, hence for  $\varepsilon$  sufficiently small the expression is less than one, proving the required result.  $\square$

*Proof of proposition 3.5.1.* Let  $\varepsilon$  sufficiently small and  $\Phi \in \mathcal{G}(\varepsilon)$  be such that lemma 3.5.3 (2) holds, which excludes only a set of measure zero. We prove by induction on the degree of  $\Phi \in \mathcal{G}(\varepsilon)$ . Let  $\Phi \in \mathcal{T}_0 \cap \mathcal{G}(\varepsilon)$ . Then  $\tau = 0$  so by theorem 3.3.1 and theorem 3.4.17,

$$\hat{P}_0^\varepsilon(\Phi) = \zeta^\varepsilon(\Phi) f_0(x_0, v_0) = \zeta^\varepsilon(\Phi) P_0^\varepsilon(\Phi) = R_t^\varepsilon(\Phi).$$

Hence by lemma 3.5.2 and lemma 3.5.3 (1) for  $t \geq 0$ ,

$$\begin{aligned}
\hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) &\geq -2\eta_t^\varepsilon(\Phi) R_t^\varepsilon(\Phi) \int_\tau^t \exp \left( -2\eta_s^\varepsilon(\Phi) \int_s^t L_\sigma^\varepsilon(\Phi) \, d\sigma \right) L_s^\varepsilon(\Phi) \, ds \\
&\geq -\rho_t^{\varepsilon, 0}(\Phi) R_t^\varepsilon(\Phi) \\
&= -\hat{\rho}_t^\varepsilon(\Phi) R_t^\varepsilon(\Phi).
\end{aligned}$$

This proves the proposition in the base case. Now suppose that the proposition holds for all histories in  $\mathcal{T}_{j-1} \cap \mathcal{G}(\varepsilon)$  for some  $j \geq 1$  and let  $\Phi \in \mathcal{T}_j \cap \mathcal{G}(\varepsilon)$ . For  $t < \tau$  the proposition holds trivially since the left hand side is 0. Consider  $t \geq \tau$ . By theorem 3.3.1 and theorem 3.4.17 we have,

$$\hat{P}_\tau^\varepsilon(\Phi) = \left( \frac{1 - \gamma^\varepsilon(t)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) \, d\bar{x} \, d\bar{v}} \right) \hat{P}_\tau^\varepsilon(\bar{\Phi}) g_\tau(x(\tau) + \varepsilon\nu, v') [(v(\tau^-) - v') \cdot \nu]_+,$$

and

$$R_\tau^\varepsilon(\Phi) = R_\tau^\varepsilon(\bar{\Phi}) g_\tau(x(\tau) + \varepsilon\nu, v') [(v(\tau^-) - v') \cdot \nu]_+.$$

Since  $\bar{\Phi} \in \mathcal{T}_{j-1} \cap \mathcal{G}(\varepsilon)$  by the inductive assumption we have that, for  $\tau = \tau(\Phi)$ ,

$$\hat{P}_\tau^\varepsilon(\bar{\Phi}) \geq R_\tau^\varepsilon(\bar{\Phi}) - \hat{\rho}_\tau^\varepsilon(\bar{\Phi}) R_\tau^\varepsilon(\bar{\Phi}).$$

Hence by lemma 3.5.3 (2) for  $\varepsilon$  sufficiently small,

$$\begin{aligned} \hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi) &= g_\tau(x(\tau) + \varepsilon\nu, v')[(v(\tau^-) - v') \cdot \nu]_+ \\ &\quad \left( \frac{1 - \gamma^\varepsilon(t)}{\int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v}) \mathbb{1}_\tau^\varepsilon[\Phi](\bar{x}, \bar{v}) d\bar{x} d\bar{v}} \hat{P}_\tau^\varepsilon(\bar{\Phi}) - R_\tau^\varepsilon(\bar{\Phi}) \right) \\ &\geq g_\tau(x(\tau) + \varepsilon\nu, v')[(v(\tau^-) - v') \cdot \nu]_+ \left( (1 - \varepsilon) \hat{P}_\tau^\varepsilon(\bar{\Phi}) - R_\tau^\varepsilon(\bar{\Phi}) \right) \\ &\geq g_\tau(x(\tau) + \varepsilon\nu, v')[(v(\tau^-) - v') \cdot \nu]_+ \\ &\quad \left( (1 - \varepsilon) (R_\tau^\varepsilon(\bar{\Phi}) - \hat{\rho}_\tau^\varepsilon(\bar{\Phi}) R_\tau^\varepsilon(\bar{\Phi})) - R_\tau^\varepsilon(\bar{\Phi}) \right) \\ &= g_\tau(x(\tau) + \varepsilon\nu, v')[(v(\tau^-) - v') \cdot \nu]_+ R_\tau^\varepsilon(\bar{\Phi}) \left( (1 - \varepsilon) (1 - \hat{\rho}_\tau^\varepsilon(\bar{\Phi})) - 1 \right) \\ &= R_\tau^\varepsilon(\Phi) (-\varepsilon - (1 - \varepsilon) \hat{\rho}_\tau^\varepsilon(\bar{\Phi})). \end{aligned} \tag{3.5.8}$$

Now the trajectory of the tagged particle up to time  $\tau$  is identical for  $\Phi$  and  $\bar{\Phi}$  and recalling that  $\eta_t^\varepsilon(\Phi)$  is non-decreasing with  $t$  it follows,

$$\eta_\tau^\varepsilon(\bar{\Phi}) = \eta_\tau^\varepsilon(\Phi) \leq \eta_t^\varepsilon(\Phi).$$

Further by (3.5.1) it follows that  $C^\varepsilon(\bar{\Phi}) \leq C^\varepsilon(\Phi)$ . These imply that,

$$\hat{\rho}_\tau^\varepsilon(\bar{\Phi}) = \rho_\tau^{\varepsilon, j-1}(\bar{\Phi}) \leq \rho_\tau^{\varepsilon, j-1}(\Phi).$$

Hence (3.5.8) becomes,

$$\begin{aligned} \hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi) &\geq -R_\tau^\varepsilon(\Phi) (\varepsilon + (1 - \varepsilon) \hat{\rho}_\tau^\varepsilon(\bar{\Phi})) \\ &\geq -R_\tau^\varepsilon(\Phi) (\varepsilon + (1 - \varepsilon) \rho_\tau^{\varepsilon, j-1}(\Phi)). \end{aligned}$$

Using (3.5.7) and that  $L_s^\varepsilon(\Phi)$  is non-negative, this gives that,

$$\begin{aligned} &\exp \left( \int_\tau^t -(1 + 2\eta_s^\varepsilon(\Phi)) L_s^\varepsilon(\Phi) ds \right) (\hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi)) \\ &\geq -\exp \left( - \int_\tau^t (1 + 2\eta_s^\varepsilon(\Phi)) L_s^\varepsilon(\Phi) ds \right) R_\tau^\varepsilon(\Phi) (\varepsilon + (1 - \varepsilon) \rho_\tau^{\varepsilon, j-1}(\Phi)) \\ &\geq -R_t^\varepsilon(\Phi) \exp \left( - \int_\tau^t 2\eta_s^\varepsilon(\Phi) L_s^\varepsilon(\Phi) ds \right) (\varepsilon + (1 - \varepsilon) \rho_\tau^{\varepsilon, j-1}(\Phi)) \\ &\geq -R_t^\varepsilon(\Phi) (\varepsilon + (1 - \varepsilon) \rho_\tau^{\varepsilon, j-1}(\Phi)). \end{aligned}$$

Finally we use lemma 3.5.2, lemma 3.5.3 (1) and (3.5.2) to see that,

$$\begin{aligned}
\hat{P}_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) &\geq \exp\left(-\int_\tau^t (1 + 2\eta_s^\varepsilon(\Phi))L_s^\varepsilon(\Phi) \, ds\right) (\hat{P}_\tau^\varepsilon(\Phi) - R_\tau^\varepsilon(\Phi)) \\
&\quad - 2\eta_t^\varepsilon(\Phi)R_t^\varepsilon(\Phi) \int_\tau^t \exp\left(-2\eta_s^\varepsilon(\Phi) \int_s^t L_\sigma^\varepsilon(\Phi) \, d\sigma\right) L_s^\varepsilon(\Phi) \, ds \\
&\geq -R_t^\varepsilon(\Phi)(\varepsilon + (1 - \varepsilon)\rho_\tau^{\varepsilon,j-1}(\Phi)) - \rho_t^{\varepsilon,0}(\Phi)R_t^\varepsilon(\Phi) \\
&= -R_t^\varepsilon(\Phi)\left(\varepsilon + (1 - \varepsilon)\rho_\tau^{\varepsilon,j-1}(\Phi) + \rho_t^{\varepsilon,0}(\Phi)\right) \\
&= -R_t^\varepsilon(\Phi)\rho_t^{\varepsilon,j}(\Phi) = -R_t^\varepsilon(\Phi)\hat{\rho}_t^\varepsilon(\Phi).
\end{aligned}$$

This proves the inductive step and so completes the proof of the proposition.  $\square$

### 3.5.2 Convergence between $P_t^0$ and $\hat{P}_t^\varepsilon$ and the proof of theorem 3.2.4

**Lemma 3.5.4.** *For any  $\delta > 0$  there exists an  $\varepsilon' > 0$  such that for any  $0 < \varepsilon < \varepsilon'$ , any  $t \in [0, T]$  and almost all  $\Phi \in \mathcal{G}(\varepsilon)$ ,*

$$\hat{\rho}_t^\varepsilon(\Phi) < \delta.$$

*Proof.* Fix  $\delta > 0$ . By (3.5.5) we have for  $\Phi \in \mathcal{G}(\varepsilon)$ ,

$$\begin{aligned}
\eta_t^\varepsilon(\Phi) &= \int_{U \times \mathbb{R}^3} g_0(\bar{x}, \bar{v})(1 - \mathbb{1}_t^\varepsilon[\Phi](\bar{x}, \bar{v})) \, d\bar{x} \, d\bar{v} \leq \pi\varepsilon^2 T M_g(V(\varepsilon) + 1) \\
&=: C_1 \varepsilon^2 (1 + V(\varepsilon)).
\end{aligned}$$

Secondly we note that for almost all  $\Phi \in \mathcal{G}(\varepsilon)$  and any  $t \in [0, T]$ ,

$$\begin{aligned}
\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon\nu, \bar{v})[(v(t) - \bar{v}) \cdot \nu]_+ &\leq \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \bar{g}(\bar{v})(|v(t)| + |\bar{v}|) \, d\bar{v} \, d\nu \\
&\leq \pi \int_{\mathbb{R}^3} \bar{g}(\bar{v})(V(\varepsilon) + |\bar{v}|) \, d\bar{v} \\
&\leq \pi M_g(1 + V(\varepsilon)) =: C_2(1 + V(\varepsilon)).
\end{aligned}$$

Hence,

$$C^\varepsilon(\Phi) = 2 \sup_{t \in [0, T]} \left\{ \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_t(x(t) + \varepsilon\nu, \bar{v})[(v(t) - \bar{v}) \cdot \nu]_+ \right\} \leq 2C_2(1 + V(\varepsilon)).$$

This implies that,

$$\rho_t^{\varepsilon,0}(\Phi) = \eta_t^\varepsilon(\Phi)C^\varepsilon(\Phi)t \leq 2C_1C_2T\varepsilon^2(1 + V(\varepsilon))^2.$$

Using (3.4.4) there exists an  $\varepsilon_1 > 0$  such that for  $\varepsilon < \varepsilon_1$  we have,

$$\rho_t^{\varepsilon,0}(\Phi) < \delta/3.$$

Further by (3.4.4) there exists  $\varepsilon_2 > 0$  such that for  $\varepsilon < \varepsilon_2$ ,

$$\rho_t^{\varepsilon,0}(\Phi)M(\varepsilon) \leq 2C_1C_2T\varepsilon^2(1+V(\varepsilon))^2M(\varepsilon) < \delta/3.$$

And again by (3.4.4) there exists  $\varepsilon_3 > 0$  such that for  $\varepsilon < \varepsilon_3$ ,

$$\varepsilon M(\varepsilon) < \delta/3.$$

Take  $\varepsilon' = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, 1\}$ . Then for any  $0 < \varepsilon < \varepsilon'$  and for almost all  $\Phi \in \mathcal{G}(\varepsilon)$  we have by (3.5.3),

$$\begin{aligned} \hat{\rho}_t^\varepsilon(\Phi) &= \rho_t^{n(\Phi),\varepsilon}(\Phi) = (1-\varepsilon)^{n(\Phi)}\rho_t^{\varepsilon,0}(\Phi) + (\rho_t^{\varepsilon,0}(\Phi) + \varepsilon) \sum_{j=1}^{n(\Phi)} (1-\varepsilon)^{n(\Phi)-j} \\ &\leq \rho_t^{\varepsilon,0}(\Phi) + (\rho_t^{\varepsilon,0}(\Phi) + \varepsilon) \times n(\Phi) \leq \rho_t^{\varepsilon,0}(\Phi) + \rho_t^{\varepsilon,0}(\Phi)M(\varepsilon) + \varepsilon M(\varepsilon) < \delta. \end{aligned}$$

Proving the required result. □

**Proposition 3.5.5.** *Uniformly for  $t \in [0, T]$ ,*

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{MT} \setminus \mathcal{G}(\varepsilon)} P_t^0(\Phi) d\Phi = 0.$$

*Proof.* We first show that

$$\int_{\mathcal{MT} \setminus \mathcal{G}(0)} P_t^0(\Phi) d\Phi = 0. \tag{3.5.9}$$

To this aim note that  $\mathcal{T}_0 \setminus R(0)$  is empty since histories with zero collisions cannot include a re-collision. Let  $\Phi \in \mathcal{T}_1 \setminus R(0)$  and denote  $\Phi = ((x_0, v_0), (\tau, \nu, v'))$ . Then the initial position and velocity of the background particle is  $(x_0 + \tau(v_0 - v'), v')$ . Denote the velocity of the background particle after the collision by  $\bar{v}$ . Then  $v(\tau) = v_0 - \nu(v_0 - v') \cdot \nu$  and  $\bar{v} = v' + \nu(v_0 - v') \cdot \nu$ . Note that this gives,

$$(\bar{v} - v(\tau)) \cdot \nu = (v' - v_0) \cdot \nu + 2(v_0 - v') \cdot \nu = (v_0 - v') \cdot \nu.$$

Since  $\Phi \in \mathcal{T}_1 \setminus R(0)$  the tagged particle sees the background particle again at some

time  $s \in (\tau, T]$ . Hence at that  $s$  there exists an  $m \in \mathbb{Z}^3$  such that,

$$x(s) + m = x_0 + \tau v_0 + (s - \tau)v(\tau) + m = x_0 + \tau(v_0 - v') + \tau v' + (s - \tau)\bar{v}.$$

Which gives,

$$(s - \tau)v(\tau) + m = (s - \tau)\bar{v}.$$

Hence,

$$\frac{m}{s - \tau} = \bar{v} - v(\tau)$$

This implies

$$\frac{m \cdot \nu}{s - \tau} = (v_0 - v') \cdot \nu.$$

Hence if we consider  $v_0, \nu$  and  $v'$  fixed, then  $\tau$  must be in a countable set hence  $\mathcal{T}_1 \setminus R(0)$  is a set of zero measure.

Now let  $j \geq 2$  and consider  $\Phi \in \mathcal{T}_j \setminus R(0)$ . Then either two of the collisions in  $\Phi$  are with the same background particle, or the tagged particle will collide with one of the background particles again for some time  $s \in (\tau, T]$ . Let  $\Phi = ((x_0, v_0), (t_1, \nu_1, v_1), \dots, (t_j, \nu_j, v_j))$ . If we are in the first case there exists an  $l \leq j$  and a  $k < l$  such that the  $k$ th and  $l$ th collision are with the same background particle. Hence,

$$v_l = v_k + \nu_l(v(t_l^-) - v_k) \cdot \nu_l,$$

Thus  $v_l$  is determined by  $v_k, \nu_l$  and  $v(t_l^-)$ , so  $v_l$  can only be in a set of zero measure. In the second case, there exists a  $1 \leq k \leq n$ , an  $s \in (\tau, T]$  and a  $m \in \mathbb{Z}^3$  such that

$$x(s) + m = x_k(s). \tag{3.5.10}$$

We prove that this implies that  $t_k$  is in a set of zero measure. Note,

$$x(s) = x_0 + t_1 v_0 + (t_2 - t_1)v(t_1) + \dots + (t_j - t_{j-1})v(t_{j-1}) + (s - t_j)v(t_j).$$

And if we denote the velocity of background particle  $k$  after its collision at  $t_k$  as  $\bar{v}$ ,

$$\begin{aligned} x_k(s) &= x_k(t_k) + (s - t_k)\bar{v} \\ &= x(t_k) + (s - t_k)\bar{v} \\ &= x_0 + t_1 v_0 + (t_2 - t_1)v(t_1) + \dots + (t_k - t_{k-1})v(t_{k-1}) + (s - t_k)\bar{v}. \end{aligned}$$



Then (3.5.10) gives,

$$m + (t_{k+1} - t_k)v(t_k) + \cdots + (t_j - t_{j-1})v(t_{j-1}) + (s - t_j)v(t_j) = (s - t_k)\bar{v}.$$

Rearranging and taking the dot product with  $\nu_k$  gives that,

$$\begin{aligned} & t_k(v(t_k) - \bar{v}) \cdot \nu_k \\ &= m \cdot \nu_k + \left( t_{k+1}v(t_k) + \cdots + (t_j - t_{j-1})v(t_{j-1}) + (s - t_j)v(t_j) - s\bar{v} \right) \cdot \nu_k. \end{aligned} \tag{3.5.11}$$

Since

$$(v(t_k) - \bar{v}) \cdot \nu_k = (v_k - v(t_{k-1})) \cdot \nu_k \neq 0,$$

and  $v(t_k)$  does not depend on  $t_k$  (in the sense that  $v(t_k)$  is the same for any  $t_k \in (t_{k-1}, t_{k+1})$ ) it follows that  $t_k \in (t_{k-1}, t_{k+1})$  must be in the countable set defined by (3.5.11). Therefore  $\mathcal{T}_j \setminus R(0)$  is a set of zero measure. Since  $\mathcal{MT} = \cup_{j \geq 0} \mathcal{T}_j$  it follows that,

$$\int_{\mathcal{MT} \setminus R(0)} P_t^0(\Phi) d\Phi = 0.$$

The other conditions on  $\mathcal{G}(0)$  are clear so (3.5.9) holds. Now  $\mathcal{G}(\varepsilon)$  is increasing as  $\varepsilon$  decreases so for any  $\Phi \in \mathcal{MT}$ ,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{1}\{\Phi \in \mathcal{G}(\varepsilon)\} = \mathbb{1}\{\Phi \in \mathcal{G}(0)\} \leq 1.$$

By the dominated convergence theorem, since  $P_t^0$  is a probability measure,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{MT} \setminus \mathcal{G}(\varepsilon)} P_t^0(\Phi) d\Phi &= \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{MT}} P_t^0(\Phi) \mathbb{1}\{\Phi \notin \mathcal{G}(\varepsilon)\} d\Phi \\ &= \int_{\mathcal{MT}} P_t^0(\Phi) \mathbb{1}\{\Phi \notin \mathcal{G}(0)\} d\Phi = 0. \end{aligned}$$

□

We can now prove the convergence between  $P_t^0$  and  $\hat{P}_t^\varepsilon$ , which will then be used to prove theorem 3.2.4.

**Theorem 3.5.6.** *Uniformly for  $t \in [0, T]$ ,*

$$\lim_{\varepsilon \rightarrow 0} \sup_{S \subset \mathcal{MT}} \left| \int_S P_t^0(\Phi) - \hat{P}_t^\varepsilon(\Phi) d\Phi \right| = 0.$$

*Proof.* Let  $\delta > 0$  and  $S \subset \mathcal{MT}$ . By proposition 3.5.5, for  $\varepsilon$  sufficiently small,

$$\int_{S \setminus \mathcal{G}(\varepsilon)} P_t^0(\Phi) d\Phi \leq \int_{\mathcal{MT} \setminus \mathcal{G}(\varepsilon)} P_t^0(\Phi) d\Phi < \frac{\delta}{4}.$$

By theorem 3.3.1 for  $\varepsilon$  sufficiently small,

$$\int_{S \cap \mathcal{G}(\varepsilon)} |P_t^0(\Phi) - P_t^\varepsilon(\Phi)| d\Phi \leq \int_{\mathcal{MT}} |P_t^0(\Phi) - P_t^\varepsilon(\Phi)| d\Phi < \frac{\delta}{4}.$$

Hence,

$$\begin{aligned} \int_S P_t^0(\Phi) - \hat{P}_t^\varepsilon(\Phi) d\Phi &= \int_{S \cap \mathcal{G}(\varepsilon)} P_t^0(\Phi) - \hat{P}_t^\varepsilon(\Phi) d\Phi + \int_{S \setminus \mathcal{G}(\varepsilon)} P_t^0(\Phi) - \hat{P}_t^\varepsilon(\Phi) d\Phi \\ &< \int_{S \cap \mathcal{G}(\varepsilon)} P_t^0(\Phi) - \hat{P}_t^\varepsilon(\Phi) d\Phi + \frac{\delta}{4} \\ &= \frac{\delta}{4} + \int_{S \cap \mathcal{G}(\varepsilon)} P_t^0(\Phi) - P_t^\varepsilon(\Phi) d\Phi \\ &\quad + \int_{S \cap \mathcal{G}(\varepsilon)} P_t^\varepsilon(\Phi) - \hat{P}_t^\varepsilon(\Phi) d\Phi \\ &< \frac{\delta}{2} + \int_{S \cap \mathcal{G}(\varepsilon)} P_t^\varepsilon(\Phi) - \hat{P}_t^\varepsilon(\Phi) d\Phi. \end{aligned} \tag{3.5.12}$$

Now by the definition of  $\zeta^\varepsilon(\Phi)$  (3.4.3) we see that since  $g_0$  is a probability measure  $\zeta^\varepsilon(\Phi) \leq 1$ . Hence by lemma 3.5.4 for  $\varepsilon$  sufficiently small and almost all  $\Phi \in \mathcal{G}(\varepsilon)$ ,

$$\zeta^\varepsilon(\Phi) \hat{\rho}_t^\varepsilon(\Phi) < \frac{\delta}{4}. \tag{3.5.13}$$

Also by (3.2.3),

$$\int_{B_\varepsilon(x_0)} \int_{\mathbb{R}^3} g_0(\bar{x}, \bar{v}) d\bar{v} d\bar{x} \leq \int_{B_\varepsilon(x_0)} \int_{\mathbb{R}^3} \bar{g}(\bar{v}) d\bar{v} d\bar{x} \leq \frac{4}{3} M_g \pi \varepsilon^3.$$

Hence, recalling that in the Boltzmann-Grad scaling  $N\varepsilon^2 = 1$ , we have by the binomial inequality,

$$\begin{aligned} \zeta^\varepsilon(\Phi) &= \left( 1 - \int_{B_\varepsilon(x_0)} \int_{\mathbb{R}^3} g_0(\bar{x}, \bar{v}) d\bar{v} d\bar{x} \right)^N \\ &\geq 1 - N \int_{B_\varepsilon(x_0)} \int_{\mathbb{R}^3} g_0(\bar{x}, \bar{v}) d\bar{v} d\bar{x} \\ &\geq 1 - \frac{4}{3} M_g \pi \varepsilon. \end{aligned}$$

So for  $\varepsilon$  sufficiently small we have

$$1 - \zeta^\varepsilon(\Phi) < \frac{\delta}{4}.$$

Hence by proposition 3.5.1 and (3.5.13) we have, for  $\varepsilon$  sufficiently small and almost all  $\Phi \in \mathcal{G}(\varepsilon)$

$$\begin{aligned} P_t^\varepsilon(\Phi) - \hat{P}_t^\varepsilon(\Phi) &\leq P_t^\varepsilon(\Phi) - R_t^\varepsilon(\Phi) + \hat{\rho}_t^\varepsilon(\Phi) R_t^\varepsilon(\Phi) \\ &= P_t^\varepsilon(\Phi) - \zeta^\varepsilon(\Phi) P_t^\varepsilon(\Phi) + \zeta^\varepsilon(\Phi) \hat{\rho}_t^\varepsilon(\Phi) P_t^\varepsilon(\Phi) \\ &= (1 - \zeta^\varepsilon(\Phi)) P_t^\varepsilon(\Phi) + \zeta^\varepsilon(\Phi) \hat{\rho}_t^\varepsilon(\Phi) P_t^\varepsilon(\Phi) \\ &< \frac{\delta}{4} P_t^\varepsilon(\Phi) + \frac{\delta}{4} P_t^\varepsilon(\Phi) = \frac{\delta}{2} P_t^\varepsilon(\Phi). \end{aligned}$$

Hence for  $\varepsilon$  sufficiently small,

$$\int_{S \cap \mathcal{G}(\varepsilon)} P_t^\varepsilon(\Phi) - \hat{P}_t^\varepsilon(\Phi) \, d\Phi < \frac{\delta}{2} \int_{S \cap \mathcal{G}(\varepsilon)} P_t^\varepsilon(\Phi) \, d\Phi \leq \frac{\delta}{2} \int_{\mathcal{MT}} P_t^\varepsilon(\Phi) \, d\Phi = \frac{\delta}{2}.$$

Substituting this into (3.5.12) we see that for  $\varepsilon$  sufficiently small,

$$\int_S P_t^0(\Phi) - \hat{P}_t^\varepsilon(\Phi) \, d\Phi < \delta. \quad (3.5.14)$$

This holds for all  $S \subset \mathcal{MT}$  and hence for any  $S' \subset \mathcal{MT}$ , since  $P_t^0$  and  $\hat{P}_t^\varepsilon$  are probability measures,

$$\int_{S'} \hat{P}_t^\varepsilon(\Phi) - P_t^0(\Phi) \, d\Phi = \int_{\mathcal{MT} \setminus S'} \hat{P}_t^\varepsilon(\Phi) - P_t^0(\Phi) \, d\Phi < \delta.$$

Together with (3.5.14) this gives that for  $\varepsilon$  sufficiently small, for any  $S \subset \mathcal{MT}$  we have,

$$\left| \int_S \hat{P}_t^\varepsilon(\Phi) - P_t^0(\Phi) \, d\Phi \right| < \delta,$$

which completes the proof of the theorem.  $\square$

This now allows us to prove the main theorem of this chapter, theorem 3.2.4.

*Proof of theorem 3.2.4.* Let  $t \in [0, T]$  and  $\Omega \subset U \times \mathbb{R}^3$ . By theorem 3.3.1,

$$\int_{\Omega} f_t^0(x, v) \, dx \, dv = \int_{S_t(\Omega)} P_t^0(\Phi) \, d\Phi.$$

By definition  $\hat{P}_t^\varepsilon$  satisfies,

$$\int_{\Omega} \hat{f}_t^N(x, v) \, dx \, dv = \int_{S_t(\Omega)} \hat{P}_t^\varepsilon(\Phi) \, d\Phi.$$

Let  $\delta > 0$ . By theorem 3.5.6, for  $\varepsilon$  sufficiently small (or equivalently by the Boltzmann-Grad scaling,  $N\varepsilon^2 = 1$ , for  $N$  sufficiently large) and independent of  $t$ ,

$$\sup_{S \subset \mathcal{MT}} \left| \int_S P_t^0(\Phi) - \hat{P}_t^\varepsilon(\Phi) \, d\Phi \right| < \delta.$$

Hence,

$$\begin{aligned} \left| \int_{\Omega} \hat{f}_t^N(x, v) - f_t^0(x, v) \, dx \, dv \right| &= \left| \int_{S_t(\Omega)} P_t^0(\Phi) - \hat{P}_t^\varepsilon(\Phi) \, d\Phi \right| \\ &\leq \sup_{S \subset \mathcal{MT}} \left| \int_S P_t^0(\Phi) - \hat{P}_t^\varepsilon(\Phi) \, d\Phi \right| < \delta, t \end{aligned}$$

which proves the result. □

# Bibliography

- [1] Grégoire Allaire, Xavier Blanc, Bruno Despres, and Golse François. Transport et diffusion. <http://www.cmap.polytechnique.fr/~allaire/map567/M1TranspDiff.pdf>, 2015. [Online; accessed 19-July-2017].
- [2] Wolfgang Arendt, Charles J. K. Batty, Matthias Hieber, and Frank Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, volume 96 of *Monographs in Mathematics*. Birkhäuser/Springer Basel AG, Basel, second edition, 2011.
- [3] Luisa Arlotti and Bertrand Lods. Integral representation of the linear Boltzmann operator for granular gas dynamics with applications. *Journal of Statistical Physics*, 129(3):517–536, 2007.
- [4] Luisa Arlotti, Bertrand Lods, and Mustapha Mokhtar-Kharroubi. Non-autonomous honesty theory in abstract state spaces with applications to linear kinetic equations. *Commun. Pure Appl. Anal.*, 13(2):729–771, 2014.
- [5] Nathalie Ayi. From Newton’s law to the linear Boltzmann equation without cut-off. *Comm. Math. Phys.*, 350(3):1219–1274, 2017.
- [6] Jacek Banasiak and Luisa Arlotti. *Perturbations of positive semigroups with applications*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2006.
- [7] Patrick Billingsley. *Probability and measure*. Wiley Series in Probability and Statistics. John Wiley & Sons, Inc., Hoboken, NJ, 2012.
- [8] Marzia Bisi, José A Cañizo, and Bertrand Lods. Entropy dissipation estimates for the linear Boltzmann operator. *Journal of Functional Analysis*, 269(4):1028–1069, 2015.
- [9] Thierry Bodineau, Isabelle Gallagher, and Laure Saint-Raymond. The Brownian motion as the limit of a deterministic system of hard-spheres. *Inventiones mathematicae*, 203(2):493–553, 2015.

- [10] Thierry Bodineau, Isabelle Gallagher, and Laure Saint-Raymond. From hard sphere dynamics to the Stokes–Fourier equations: An  $L^2$  analysis of the Boltzmann–Grad limit. *Annals of PDE*, 3(1):2, Feb 2017.
- [11] Carlo Boldrighini, Leonid A. Bunimovich, and Yakov. G. Sinai. On the Boltzmann equation for the Lorentz gas. *J. Statist. Phys.*, 32(3):477–501, 1983.
- [12] Torsten Carleman. *Problemes mathématiques dans la théorie cinétique de gaz*, volume 2. Almqvist & Wiksells boktr, 1957.
- [13] Carlo Cercignani. *The Boltzmann equation and its applications*, volume 67 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1988.
- [14] Carlo Cercignani, Reinhard Illner, and Mario Pulvirenti. *The mathematical theory of dilute gases*, volume 106 of *Applied Mathematical Sciences*. Springer-Verlag, 1994.
- [15] Donald L. Cohn. *Measure theory*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, New York, second edition, 2013.
- [16] Robert Dautray and Jacques-Louis Lions. *Mathematical analysis and numerical methods for science and technology. Vol. 6*. Springer-Verlag, Berlin, 1993. Evolution problems. II, With the collaboration of Claude Bardos, Michel Cessenat, Alain Kavenoky, Patrick Lascaux, Bertrand Mercier, Olivier Pironneau, Bruno Scheurer and Rémi Sentis, Translated from the French by Alan Craig.
- [17] L. Desvillettes and V. Ricci. The Boltzmann-Grad limit of a stochastic Lorentz gas in a force field. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 2(2):637–648, 2007.
- [18] Gabriella Di Blasio. Differentiability of spatially homogeneous solutions of the Boltzmann equation in the non Maxwellian case. *Comm. Math. Phys.*, 38:331–340, 1974.
- [19] S. Friedlander and D. Serre, editors. *Handbook of mathematical fluid dynamics. Vol. I*. North-Holland, Amsterdam, 2002.
- [20] Isabelle Gallagher, Laure Saint-Raymond, and Benjamin Texier. *From Newton to Boltzmann: hard spheres and short-range potentials*. Zurich Lectures in Advanced Mathematics. European Mathematical Society, 2013.
- [21] Giovanni Gallavotti. Divergences and the approach to equilibrium in the Lorentz and the wind-tree models. *Physical Review*, 185(1):308, 1969.

- [22] Giovanni Gallavotti. *Statistical Mechanics. A Short Treatise*. Theoretical and Mathematical Physics. Springer-Verlag Berlin Heidelberg, 1999.
- [23] François Golse. On the periodic Lorentz gas and the Lorentz kinetic equation. *Ann. Fac. Sci. Toulouse Math. (6)*, 17(4):735–749, 2008.
- [24] R. Illner and M. Pulvirenti. Global validity of the Boltzmann equation for two- and three-dimensional rare gas in vacuum. Erratum and improved result. *Comm. Math. Phys.*, 121(1):143–146, 1989.
- [25] Reinhard Illner and Mario Pulvirenti. Global validity of the Boltzmann equation for a two-dimensional rare gas in vacuum. *Comm. Math. Phys.*, 105(2):189–203, 1986.
- [26] Francis Gordon King. *BBGKY hierarchy for positive potentials*. ProQuest LLC, Ann Arbor, MI, 1975. Thesis (Ph.D.)—University of California, Berkeley.
- [27] Oscar E. Lanford. *Dynamical Systems, Theory and Applications: Battelle Seattle 1974 Rencontres*, chapter Time evolution of large classical systems, pages 1–111. Springer Berlin Heidelberg, Berlin, Heidelberg, 1975.
- [28] Joel L. Lebowitz and Herbert Spohn. Transport properties of the Lorentz gas: Fourier’s law. *J. Statist. Phys.*, 19(6):633–654, 1978.
- [29] Joel L. Lebowitz and Herbert Spohn. Microscopic basis for Fick’s law for self-diffusion. *J. Statist. Phys.*, 28(3):539–556, 1982.
- [30] Joel L. Lebowitz and Herbert Spohn. Steady state self-diffusion at low density. *J. Statist. Phys.*, 29(1):39–55, 1982.
- [31] Hendrik Lorentz. The motion of electrons in metallic bodies i. *Koninklijke Nederlandsche Akademie van Wetenschappen Proceedings*, 7:438–453, 1905.
- [32] Jens Marklof. Kinetic transport in crystals. In *XVIth International Congress on Mathematical Physics*, pages 162–179. World Sci. Publ., Hackensack, NJ, 2010.
- [33] Jens Marklof and Andreas Strömbergsson. The Boltzmann-Grad limit of the periodic Lorentz gas. *Ann. of Math. (2)*, 174(1):225–298, 2011.
- [34] K. Matthies and G. Stone. Derivation of a Nonautonomous Linear Boltzmann Equation from a Heterogeneous Rayleigh Gas. <https://arxiv.org/abs/1706.03532>, June 2017. Preprint.

- [35] Karsten Matthies, George Stone, and Florian Theil. The derivation of the linear boltzmann equation from a rayleigh gas particle model. *Kinetic and Related Models*, 11(2):8–8, 2018.
- [36] Karsten Matthies and Florian Theil. Validity and non-validity of propagation of chaos. *Analysis and Stochastics of growth processes and Interface Models*, pages 101–119, 2008.
- [37] Karsten Matthies and Florian Theil. Validity and failure of the Boltzmann approximation of kinetic annihilation. *Journal of nonlinear science*, 20(1):1–46, 2010.
- [38] Karsten Matthies and Florian Theil. A semigroup approach to the justification of kinetic theory. *SIAM Journal on Mathematical Analysis*, 44(6):4345–4379, 2012.
- [39] Rainer Nagel and Gregor Nickel. Well-posedness for nonautonomous abstract Cauchy problems. In *Evolution equations, semigroups and functional analysis (Milano, 2000)*, volume 50 of *Progr. Nonlinear Differential Equations Appl.*, pages 279–293. Birkhäuser, Basel, 2002.
- [40] Amnon Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, 1983.
- [41] M. Pulvirenti. Global validity of the Boltzmann equation for a three-dimensional rare gas in vacuum. *Comm. Math. Phys.*, 113(1):79–85, 1987.
- [42] Mario Pulvirenti, Chiara Saffirio, and Sergio Simonella. On the validity of the Boltzmann equation for short range potentials. *Reviews in Mathematical Physics*, 26(02), 2014.
- [43] Laure Saint-Raymond. Some recent results about the sixth problem of Hilbert. In *Analysis and simulation of fluid dynamics*, Adv. Math. Fluid Mech., pages 183–199. Birkhäuser, Basel, 2007.
- [44] Laure Saint-Raymond. *Hydrodynamic limits of the Boltzmann equation*, volume 1971 of *Lecture Notes in Mathematics*. Springer-Verlag Berlin Heidelberg, 2009.
- [45] Herbert Spohn. The Lorentz process converges to a random flight process. *Comm. Math. Phys.*, 60(3):277–290, 1978.
- [46] Herbert Spohn. Kinetic equations from Hamiltonian dynamics: the Markovian approximations. In *Kinetic theory and gas dynamics*, volume 293 of *CISM Courses and Lect.*, pages 183–211. Springer, Vienna, 1988.



- [47] Herbert Spohn. *Large Scale Dynamics of Interacting Particles*. Texts and Monographs in Physics. Springer Berlin Heidelberg, 1991.
- [48] Seiji Ukai. On the existence of global solutions of mixed problem for non-linear Boltzmann equation. *Proc. Japan Acad.*, 50:179–184, 1974.
- [49] Seiji Ukai. Les solutions globales de l'équation de Boltzmann dans l'espace tout entier et dans le demi-espace. *C. R. Acad. Sci. Paris Sér. A-B*, 282(6):Ai, A317–A320, 1976.
- [50] Seiji Ukai. The Boltzmann-Grad limit and Cauchy-Kovalevskaya theorem. *Japan journal of industrial and applied mathematics*, 18(2):383–392, 2001.
- [51] Henk van Beijeren, Oscar E. Lanford, III, Joel L. Lebowitz, and Herbert Spohn. Equilibrium time correlation functions in the low-density limit. *J. Statist. Phys.*, 22(2):237–257, 1980.